Random measurements are almost maximally incompatible

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Measurement incompatibility

Quantum states and measurements

- Motivation: Classical state $\rightsquigarrow$ probability distributions: $p \in \mathbb{R}^d$, $p \geq 0$, $\sum_i p_i = 1$
- Quantum states $\rightsquigarrow$ density matrices: $\rho \in M_d(\mathbb{C})$, $\rho \geq 0$, $\operatorname{Tr} \rho = 1$
- Measurement outcomes are labeled $\{1, \ldots, k\}$, need to be assigned probabilities
- Measurements: Tuples of matrices $(E_1, \ldots, E_k)$ such that $(\operatorname{Tr}[E_1 \rho], \ldots, \operatorname{Tr}[E_k \rho])$ is a probability distribution for all states ρ
  - $\operatorname{Tr}[E_i \rho] \in \mathbb{R} \rightsquigarrow E_i = E_i^*$
  - $\operatorname{Tr}[E_i \rho] \geq 0 \rightsquigarrow E_i \geq 0$
  - $\sum_i \operatorname{Tr}[E_i \rho] = 1 \rightsquigarrow \sum_i E_i = I_d$
- Tuples of PSD matrices summing to identity are called positive operator-valued measures (POVMs).

Quantum measurements: Compatibility

- Quantum measurements $\rightsquigarrow$ give the probabilities of the classical outcomes when a quantum state enters a measurement apparatus. Mathematically, measurements are modeled by POVMs.

Definition

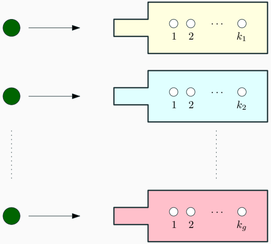
Two POVMs, $E = (E_1, \ldots, E_k)$ and $F = (F_1, \ldots, F_l)$, are called compatible if there exists a third POVM $C = (C_{ij})_{i \in [k], j \in [l]}$ such that

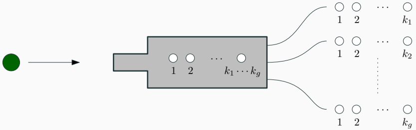
$$\forall i \in [k], \quad E_i = \sum_{j=1}^{l} C_{ij} \qquad \text{and} \qquad \forall j \in [l], \quad F_j = \sum_{i=1}^{k} C_{ij}$$

The definition generalizes to g-tuples of POVMs $E^{(1)}, \ldots, E^{(g)}$, having respectively $k_1, \ldots k_g$ outcomes, where the joint POVM C has outcome set $[k_1] \times \cdots \times [k_g]$.

- Other way to say that: jointly measurable

What does it mean?





- Compatible measurements can be simulated by a single joint measurement, by classically post-processing its outputs: $E_{i|x} = \sum_{\lambda \in \Lambda} p_{i|\lambda,x} C_\lambda \quad \forall i \in [k_x], \forall x \in [g]$.
- Examples:
  1. Trivial POVMs $E = (p_i I_d)$ and $F = (q_j I_d)$ are compatible.
  2. Commuting POVMs $[E_i, F_j] = 0$ are compatible.
  3. If the POVM E is projective, then E and F are compatible iff they commute.

Noisy POVMs

- POVMs can be made compatible by adding noise, i.e. mixing in trivial POVMs.
- There are different types of noise, we focus on white or uniform noise:

Definition

Let $k \in \mathbb{N}$ and let $(E_i)_{i\in[k]}$ be a POVM. Let $t \in [0,1]$ be a noise parameter. Then, we define the noisy POVM $(E_i^{(t)})_{i\in[k]}$ as

$$E_i^{(t)} := tE_i + (1-t)\frac{I}{k} \qquad \forall i \in [k].$$

Note that $E_i^{(1)} = E_i$ and that $E_i^{(0)} = I/k$.

- Taking $t = 1/2$ suffices to render any pair of dichotomic POVMs compatible. To see this, choose $C_{ij} := (E_i + F_j)/4$

The compatibility degree

Definition (Compatibility degree of a set of measurements)

Let $d, g \in \mathbb{N}$ and $k_x \in \mathbb{N}$ for all $x \in [g]$. Let $E := \{(E_{i|x})_{i\in[k_x]}\}_{x\in[g]}$ be a set of POVMs. Then, their compatibility degree is defined as

$$\tau(E) := \max\left\{ t \in [0,1] : \left\{ \left(E_{i|x}^{(t)} \right)_{i\in[k_x]} \right\}_{x\in[g]} \text{ is a set of compatible measurements} \right\}.$$

In words: $1 - \tau(E)$ is the minimal amount of noise that makes the measurements in E compatible.

The minimal compatibility degree

Definition (Minimal compatibility degree)

Let $d, g \in \mathbb{N}$ and $k_x \in \mathbb{N}$ for all $x \in [g]$. Then, the minimal compatibility degree is defined as

$$\tau(d, g, (k_1, \dots, k_g)) := \min_E \tau(E),$$

where the minimum is taken over all sets E of g POVMs in dimension d, where the x-th POVM has k_x outcomes for all $x \in [g]$.

In words: $1 - \tau(d, g, (k_1, \dots, k_g))$ is the minimal amount of noise that makes any collection of g measurements on a d-dimensional quantum system where the i-th measurement has k_i outcomes compatible.

Some previous results

In [BN18], we have shown that for dichotomic measurements

$$\tau(d, g, (2, \ldots, 2)) \geq \frac{1}{\sqrt{g}}$$

and

$$\tau(d, g, (2, \ldots, 2)) = \frac{1}{\sqrt{g}} \quad \text{whenever} \quad d \geq 2^{\lceil (g-1)/2 \rceil}.$$

The quantity $\tau(d, g, (2, \ldots, 2))$ is the minimal amount of compatibility, so a worst case bound.

Main question

How incompatible are measurements generically, i.e., if we look at random measurements?

Types of random measurements considered

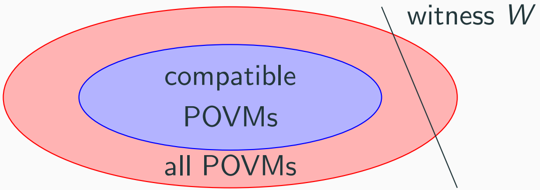
Random POVM model	Number of outcomes	Parameters
Dichotomic PVM	2	rank of the projections
Basis PVM	d	—
Induced POVM	k	dimension of the ancilla space

Types of results:

- Asymptotic: Statements which hold with probability 1 in the limit where the underlying dimension d is infinite (tools from free probability)
- Non-asymptotic: Statements which hold with probability close to 1 when the underlying dimension d is finite but large (tools from random matrix theory)

Incompatibility witnesses

Incompatibility witnesses



Definition

Let $d, g \in \mathbb{N}$ and $k_x \in \mathbb{N}$ for all $x \in [g]$. An incompatibility witness is a g-tuple of operators $W = (W_{i|x})_{i \in [k_x], x \in [g]}$, where $W_{i|x} \in M_d^{\mathrm{sa}}(\mathbb{C})$, such that

$$\max_{E \text{ compatible}} \langle W, E \rangle \leq 1.$$

Here, we have defined

$$\langle W, E \rangle := \sum_{x \in [g]} \sum_{i \in [k_x]} \operatorname{Tr}\left(W_{i|x} E_{i|x}\right) .$$

Witnesses for dichotomic measurements

- We can identify a tuple of dichotomic measurements $\{(E_x, I - E_x)\}_{x\in[g]}$ with the set $\{E_x\}$ and set $A_x := 2E_x - I$.
- Then, we can define a dichotomic incompatibility witness $W = (W_x)_{x\in[g]}$, where $W_x \in M_d^{\mathrm{sa}}(\mathbb{C})$, as

$$\max_{E \text{ compatible}} \langle W, A\rangle \leq 1\,.$$

- In [BN22], we have shown that W is an incompatibility witness if and only if there exists a quantum state $\rho \in S_d(\mathbb{C})$ such that

$$\rho - \sum_{x\in[g]} \epsilon_x W_x \geq 0 \qquad \forall \epsilon \in \{\pm 1\}^g.$$

Two random dichotomic projective measurements

Constructing random projective measurements

- Fix rank $r \in \mathbb{N}$ and let P_0 be the fixed projection

$$P_0 = \sum_{i=1}^{r} |i\rangle\langle i| \,.$$

- Then, take a Haar-random unitary U and define

$$P = UP_0U^* \,.$$

- Define the measurement to be $(P, I - P)$ and repeat the procedure g times in total with independent unitaries.
- Equivalently, we could have chosen g independent uniformly distributed subspaces of dimension r and used the projections onto them.

Two random projective measurements

Theorem

Fix $0 < \alpha, \beta < 1$ *and suppose that*

$$\left(\alpha - \frac{1}{2}\right)^2 + \left(\beta - \frac{1}{2}\right)^2 \leq \frac{1}{4}.$$

Let $E, F \subset \mathbb{C}^d$ *be independent uniformly distributed subspaces of dimensions* $\lfloor \alpha d \rfloor, \lfloor \beta d \rfloor$*, respectively. Then, for any* $0 < \epsilon \leq 1/\sqrt{2}$*,*

$$\mathbb{P}\left(\tau(P_E, P_F) \leq \frac{1}{\sqrt{2}} + \epsilon\right) \xrightarrow[d \to \infty]{} 1.$$

In this sense, two random projective measurements are close to being maximally incompatible.

Proof sketch (1/2)

- Using dichotomic incompatibility witnesses, we can argue that $\tau(P_E, P_F) \leq \tau(V^* P_E V, V^* P_F V)$ for isometries $V : \mathbb{C}^2 \hookrightarrow \mathbb{C}^d$
- We know that $\tau(\sigma_X, \sigma_Z) = \frac{1}{\sqrt{2}}$
- Thus, we are looking for isometries such that

$$V^*(2P_E - I)V = \sigma_Z \quad \text{and} \quad V^*(2P_F - I)V \xrightarrow[d\to\infty]{} \sigma_X \text{ in probability.}$$

- We call this the compression technique

Proof sketch (2/2)

- Let $\gamma = \min(\alpha, \beta, 1-\alpha, 1-\beta)$. We can always find orthonormal bases $\{e_1, \ldots, e_{\alpha d}\}$ of E and $\{f_1, \ldots, f_{\beta d}\}$ of F as well as ordered angles $0 \leq \theta_1 \leq \ldots \leq \theta_{\gamma d} \leq \pi/2$ such that, for each $i, j \in [\gamma d]$, $\langle e_i | f_j \rangle = \delta_{i,j} \cos(\theta_i)$. The θ_i's are called the (non-zero) principal angles between the subspaces E and F.
- In other words, P_E and P_F can be block diagonalized such that the blocks are either 1- or 2-dimensional. The 1D blocks have entries 0 or 1, and on the 2D blocks P_E and P_F have the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos(\theta_i)^2 & \cos(\theta_i)\sin(\theta_i) \\ \cos(\theta_i)\sin(\theta_i) & \sin(\theta_i)^2 \end{pmatrix}$$

- This form is well known from Jordan's lemma/Halmos' two projections theorem and [Aub21] proved the distribution of the θ_i for random projections and $d \to \infty$

Two random projective measurements continued

Theorem

Fix $0 < \alpha < 1$ and suppose that

$$\alpha > \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) \quad \text{or} \quad \alpha < \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right).$$

Let $E, F \subset \mathbb{C}^d$ be independent uniformly distributed subspaces of dimension $\lfloor \alpha d \rfloor$. Then, for any $0 < \epsilon \leq 2\sqrt{\lambda_\alpha}\sqrt{1-\lambda_\alpha}/(\sqrt{\lambda_\alpha} + \sqrt{1-\lambda_\alpha})$,

$$\mathbb{P}\left(\tau(P_E, P_F) \leq \frac{1}{\sqrt{\lambda_\alpha} + \sqrt{1-\lambda_\alpha}} + \epsilon\right) \xrightarrow[d\to\infty]{} 1, \text{ where } \lambda_\alpha = 4\alpha(1-\alpha),$$

and

$$\mathbb{P}\left(\tau(P_E, P_F) \geq \frac{1}{\sqrt{\lambda_\alpha} + \sqrt{1-\lambda_\alpha}}\right) \xrightarrow[d\to\infty]{} 1.$$

For α far from $1/2$, the measurements become less and less compatible.

Proof sketch

Recall the Jordan product criterion from [Hei13]:

Lemma

Let $\mathrm{M} = (M_i)_{i \in [k]}$ and $\mathrm{N} = (N_j)_{j \in [l]}$ be two POVMs on $\mathbb{C}^d$. If, for all $i \in [k]$ and $j \in [l]$,

$$M_i N_j + N_j M_i \geq 0,$$

then M and N are compatible.

We can use this lemma and [Aub21] to show that for $0 \leq t \leq \frac{1}{\sqrt{\lambda_\alpha} + \sqrt{1-\lambda_\alpha}}$, the POVMs $(P_E^{(t)}, I - P_E^{(t)})$ and $(P_F^{(t)}, I - P_F^{(t)})$ become compatible with probability going to 1 as $d \to \infty$.

Many random dichotomic projective measurements

Incompatibility of many random dichotomic projective measurements

Theorem

Let $g \leq 3d$ and let $P_1, \ldots, P_g$ be random projections of rank $d/2$. Then, with probability at least $1 - e^{-2d}$,

$$\tau(\mathrm{P}_1, \ldots, \mathrm{P}_g) \leq \frac{31}{\sqrt{g}}.$$

- Note that this bound becomes nontrivial only for g large
- Combined with the lower bound $\tau(d, g, (2, \ldots, 2)) \geq 1/\sqrt{g}$, this result implies that

$$\tau(d, g, (2, \ldots, 2)) = \Theta\left(\frac{1}{\sqrt{g}}\right) \qquad \forall d \geq \frac{g}{3},$$

while previously we only knew this to hold for $d \geq 2^{\lceil (g-1)/2 \rceil}$.

Proof sketch

- We need to show that for $\frac{31}{\sqrt{g}} \le t \le 1$, the POVMs $(P_x^{(t)}, I - P_x^{(t)})$ are incompatible.
- We need to build an incompatibility witness W such that $\langle W, A\rangle > 1$ for $A_x = 2P_x^{(t)} - I$.
- Ansatz: take $W_x = sA_x/d$
- Need to show that for $s \le c/\sqrt{g}$ and some absolute constant c

$$\frac{I}{d} - \sum_{x=1}^{g} \epsilon_x W_x \ge 0 \iff \sum_{x=1}^{g} \epsilon_x A_x \le \frac{I}{s} \qquad \forall \epsilon \in \{\pm 1\}^g,$$

- This can be done using moments of Haar random states and concentration of measure results for sums of sub-Gaussian random variables

Random basis measurements

Random basis measurements

- We consider now the case of measurements in g bases $U_1, \ldots, U_g$ of $\mathbb{C}^d$, i.e. whose effects are of the form

$$E_{i|x} = U_x |i\rangle\langle i| U_x^* \qquad \forall i \in [d], \forall x \in [g].$$

- Here, U_i are iid Haar random unitary matrices
- Equivalently, we can think of taking random unitaries U_x with columns $\psi_i^{(x)}$ and defining the POVMs to be

$$M_{i|x} = \left|\psi_i^{(x)}\middle\rangle\middle\langle\psi_i^{(x)}\right| \qquad \forall i \in [d]$$

Incompatibility of two random basis measurements

Theorem

Let $U^{(d)}, V^{(d)}$ be sequences of unitary $d \times d$ matrices, such that $U^{(d)}, V^{(d)}$ are independent, and at least one of them is Haar-distributed. Then, for all sequences

$$t_d > \frac{1}{2}\left(1 + \sqrt{\frac{3\log d}{d}}\right),$$

almost surely as $d \to \infty$, the measurements

$$\left(t_d U^{(d)} |i\rangle\langle i| U^{(d)*} + (1-t_d)\frac{I_d}{d}\right)_{i\in[d]} \quad \text{and} \quad \left(t_d V^{(d)} |i\rangle\langle i| V^{(d)*} + (1-t_d)\frac{I_d}{d}\right)_{i\in[d]}$$

are incompatible.

As for any 2 measurements $\tau(M_1, M_2) \geq 1/2$, it holds that $\tau(\mathrm{B}_1, \mathrm{B}_2) \xrightarrow[d\to\infty]{} \frac{1}{2}$. Hence, two random basis measurements are close to being maximally incompatible.

Incompatibility of many random basis measurements

Theorem

Let $g \geq 2d$ and let $B_1, \ldots, B_g$ be random basis measurements. Then, with probability at least $1 - e^{-2g\log(d)}$,

$$\tau(\mathrm{B}_1, \ldots, \mathrm{B}_g) \leq \frac{135\log(d) - 1}{d - 1}.$$

- This becomes non-trivial only for d large.
- Compatible measurements can be constructed using approximate cloning, thus

$$\tau(\mathrm{B}_1, \ldots, \mathrm{B}_g) \geq \frac{g + d}{g(d+1)}.$$

- We have shown that up to log factors this bound is tight if g is of order at least d:

$$\tau(d, g, (d, \ldots, d)) = \tilde{\Theta}\left(\frac{1}{d}\right) \qquad \forall g \geq 2d.$$

Proof sketch

- We want to construct again incompatibility witnesses of the form $W_{i|x} = \alpha(U_1, \ldots, U_g)\, U_x\, |i\rangle\langle i|\, U_x^*$
- We can prove that for
$$W_f := \sum_{x \in [g]} W_{f(x)|x},$$
if $\max_{f:[g]\to[k]} \lambda_{\max}(W_f) \le \frac{1}{d}$, then W is an incompatibility witness
- Choose $\alpha(U_1, \ldots, U_g) = \frac{1}{d \times \eta(U_1, \ldots, U_g)}$ with
$$\eta(U_1, \ldots, U_g) := \max_{f:[g]\to[d]} \lambda_{\max}\left(\sum_{x \in [g]} U_x\, |f(x)\rangle\langle f(x)|\, U_x^* \right) = \max_{\|\phi\|=1} \sum_{x \in [g]} \|U_x \phi\|_\infty^2,$$
- Using facts about Haar-random unitaries and Bernstein inequality with an ε-argument

Random induced POVMs

Constructing random induced POVMs

To obtain a random POVM on the system, we proceed as in [HJN20]:

1. Sample a random Haar-distributed isometry $V : \mathbb{C}^d \hookrightarrow \mathbb{C}^k \otimes \mathbb{C}^n$.
2. Define the POVM M on $\mathbb{C}^d$ with k outcomes by

$$M_i := V^*(|i\rangle\langle i| \otimes I_n)V \qquad \forall i \in [k].$$

- We construct a random unital, completely positive map Φ such that $M_i = \Phi(|i\rangle\langle i|)$
- This construction induces a probability measures $\nu_{d,k;n}$ on the set of k-outcome POVMs on a Hilbert space of dimension d.
- The key parameter in this model is the ratio $c := nk/d$, which quantifies the relative size of the ancilla. The value $c = 1$ corresponds to a projective measurement, while $c \to 0$ corresponds to a trivial POVM $M_i = I/k$ for each $i \in [k]$.
- Although c plays a role analogous to the noise parameter t in the white-noise models discussed previously, there is no exact correspondence between c and t.

Incompatibility of random induced POVMs

Theorem

Fix $k, g \in \mathbb{N}$ and consider a sequence of g independent random induced POVMs with k outcomes, distributed along the measure $\nu_{d,k;n_d}$, in the regime where $d \sim ckn_d$ for some fixed parameter c such that

$$c > \frac{4(k-1)g}{(k-1)^2 g^2 - 2(k-2)(k-1)g + k^2}.$$

Then, almost surely as $d \to \infty$, the g POVMs are asymptotically incompatible.

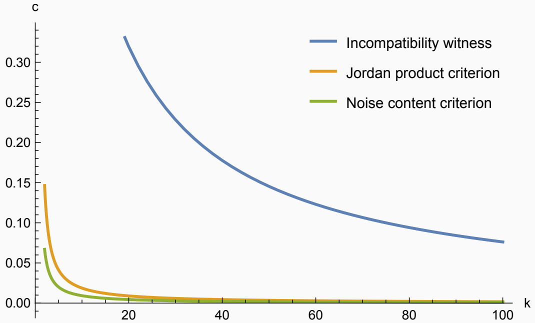
This is proved using incompatibility witnesses and asymptotic freeness of Haar-random unitary matrices and deterministic matrices, implying that $\lim_{d\to\infty} \lambda_{\max}(W_f)$ can be expressed in terms of a free convolution.

Comparison to compatibility criteria

Noise content criterion [Hei13]: The g POVMs $(M_{i|x})_{i\in[k]}$ are compatible if

$$\sum_{x=1}^{g}\sum_{i=1}^{k}\lambda_{\min}\left(M_{i|x}\right)\geq g-1.$$

For $g=2$, our bound, the noise content and the Jordan product criterion asymptotically have the same $1/k$ scaling. However:



Compatibility of random induced POVMs

Theorem

Fix $k, g \in \mathbb{N}$ and consider a sequence of g random induced POVMs with k outcomes, distributed along the measure $\nu_{d,k;n_d}$, in the regime where $d \sim ckn_d$ for some fixed parameter c such that

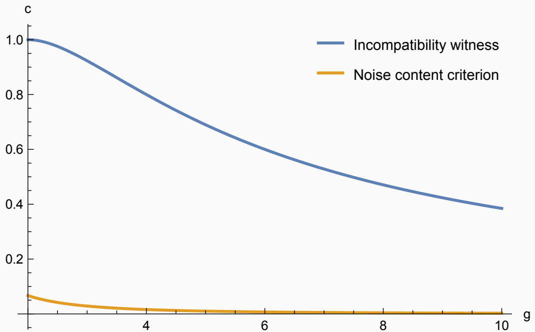
$$c > \frac{1}{g} \times \frac{1}{2g(k-1) + 2\sqrt{(g-1)(k-1)(g(k-1)+1)} - k + 2}.$$

Then, almost surely as $d \to \infty$, the g POVMs are asymptotically compatible.

This is proved using the noise content criterion and asymptotic freeness.

Comparison to compatibility criterion

For $k = 2$, our upper bound scales as $4/g$, whereas our lower bound scales as $1/(4g^2)$. Numerically:



Open questions

- Can we extend the compression technique to $g > 2$ and prove the following conjecture?

Conjecture

Given $g \geq 2$, let $P_1, \ldots, P_g$ be g independent Haar-random projections of rank $d/2$ on $\mathbb{C}^d$ and denote by $A_1, \ldots, A_g$ their associated dichotomic observables. Then, the following holds in probability:

$$\lim_{d \to \infty} \tau(A_1, \ldots, A_g) = \frac{1}{\sqrt{g}}.$$

- What are optimal incompatibility witnesses?
- Other models of randomness? Other forms on non-classicality?

Summary

- We have considered random dichotomic projective measurements, random basis measurements, and random induced POVMs
- In high-dimensional regimes and for appropriate parameter choices, random measurements are typically almost maximally incompatible
- Key techniques: incompatibility witnesses, compression technique

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Our paper: