Factorization of multimeters: a unified view on nonclassical quantum phenomena

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Introduction

- Main topic: Non-locality in general probabilistic theories (GPTs)

Main tool:

Factorization of multimeters, i.e., collections of measurements.

- GPTs are a class of operational theories in which you can talk about preparations and measurement outcomes
- Insight: Multimeters can be written as channels between GPTs

Talk outline

General probabilistic theories

Measurement incompatibility as factorizations of multimeters

Steering as factorizations of multimeters

Bell non-locality as factorizations of multimeters

Extendability of tensors

General probabilistic theories

Decomposing quantum theory

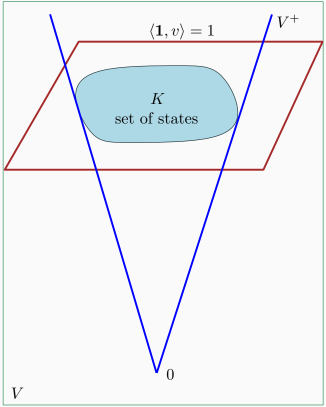
Set of quantum states:

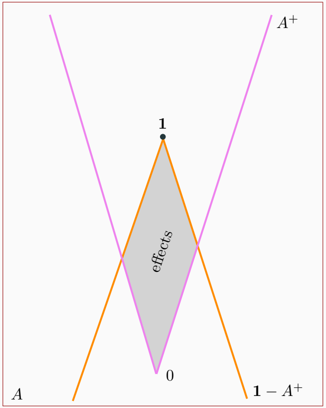
$$\mathcal{S}(\mathbb{C}^d) := \{\rho \in \mathcal{M}_d(\mathbb{C})^{\mathrm{herm}} : \rho \geq 0, \mathrm{Tr}[\rho] = 1\}\,.$$

What do we need to define a set like this?

- In which (real) vector space do my states live? $\rho \in \mathcal{M}_d(\mathbb{C})^{\mathrm{herm}}$
- What does it mean to be positive? $\rho \geq 0 \iff \rho \in \mathrm{PSD}_d$
- Which hyperplane singles out the states among the positive elements? $\mathrm{Tr}[\rho] = 1$

The parts of a GPT





- A GPT is a triple $(V, V^+, \mathbb{1})$, where V is a vector space, $V^+ \subseteq V$ is a cone, and $\mathbb{1}$ is a linear form on V; $A = V^*$, $A^+ = (V^+)^*$, and $\mathbb{1} \in A^+$
- The set of states $K := V^+ \cap \mathbb{1}^{-1}(\{1\})$

Examples of GPTs (1/2)

Example: Classical mechanics

Any *classical system* is described by the triple $\mathrm{CM}_d := (\mathbb{R}^d, \mathbb{R}^d_+, 1_d)$, $d \in \mathbb{N}$, where $1_d = (1, 1, \ldots, 1)$. The classical state space is the simplex with d vertices S_d.

Example: Quantum mechanics

Quantum theory corresponds to the triple $\mathrm{QM}_d := (\mathcal{M}_d(\mathbb{C})^{\mathrm{herm}}, \mathrm{PSD}_d, \mathrm{Tr})$, $d \in \mathbb{N}$. The quantum state space is the set of density matrices $\mathcal{S}(\mathbb{C}^d)$.

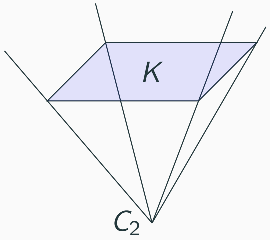
Examples of GPTs (2/2)

Example: g-bits

The *hypercube* GPT has the unit ball of ℓ_∞^n as a state space, $n \in \mathbb{N}$. It is described by the triple $\mathrm{HC}_n := (\mathbb{R}^{n+1}, C_n, \mathbb{1})$, where

$$C_n = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \; : \; x_0 \geq \max_{i \in [n]} |x_i|\}$$

$$\mathbb{1}(x_0, x_1, \dots, x_n) = x_0.$$



Bipartite systems

Let $(V_A, V_A^+, \mathbb{1}_{V_A})$ and $(V_B, V_B^+, \mathbb{1}_{V_B})$ be two GPTs. How to construct a joint GPT?

- Joint vector space $V_{AB} := V_A \otimes V_B$, order unit $\mathbb{1}_{V_{AB}} := \mathbb{1}_{V_A} \otimes \mathbb{1}_{V_B}$
- For the cone V_{AB}^+, we have in general infinitely many choices.

Definition (Minimal tensor product)

Let V_A^+ and V_B^+ be two cones. Their *minimal tensor product* is then defined as $V_A^+ \dot{\otimes} V_B^+ = \mathrm{conv}\{x_A \otimes x_B \,|\, x_A \in V_A^+, x_B \in V_B^+\}$.

Definition (Maximal tensor product)

The *maximal tensor product* of two cones is the set $V_A^+ \hat{\otimes} V_B^+ = ((V_A^+)^* \dot{\otimes} (V_B^+)^*)^*$.

We are free to choose any tensor cone, i.e., any cone V_{AB}^+ such that

$$V_A^+ \dot{\otimes} V_B^+ \subseteq V_{AB}^+ \subseteq V_A^+ \hat{\otimes} V_B^+ .$$

Examples of bipartite systems

- If one of the GPTs is CM_d, the resulting joint GPT is unique, since the tensor product of cones is unique as soon as one of them is simplicial [ALPP21]
- In quantum mechanics, $\mathrm{PSD}_m \dot{\otimes} \mathrm{PSD}_n$ are the positive semi-definite matrices that are separable and $\mathrm{PSD}_m \hat{\otimes} \mathrm{PSD}_n$ the ones that are block-positive (entanglement witnesses).
- Hence

$$\mathrm{PSD}_m \dot{\otimes} \mathrm{PSD}_n \subsetneq \mathrm{PSD}_{mn} \subsetneq \mathrm{PSD}_m \hat{\otimes} \mathrm{PSD}_n$$

- The tensor product of two copies of HC_2 with the $C_2 \hat{\otimes} C_2$ as cone gives a theory with PR boxes (correlations that win the CHSH game perfectly).

Positive maps and channels

Let $(V_A, V_A^+, \mathbb{1}_{V_A})$ and $(V_B, V_B^+, \mathbb{1}_{V_B})$ be two GPTs.

- *Channels* are linear mappings $\Phi : V_A \to V_B$ that are positive, i.e., $\Phi(V_A^+) \subseteq V_B^+$, and normalization-preserving, i.e., $\mathbb{1}_{V_B}(\Phi(v)) = \mathbb{1}_{V_A}(v)$ for all $v \in V_A$.
- We may also denote the channel $\Phi : V_A \to V_B$ simply as a map $\Phi : K_A \to K_B$ between the state spaces of the GPTs.
- Let $\{e_i\}_i$ be a basis of V_A and let $\{\alpha_j\}_j$ a dual basis of A_A, i.e., $\alpha_j(e_i) = \delta_{ij}$. We can define a special tensor $\chi_{V_A} \in A_A^+ \hat{\otimes} V_A^+$ as

$$\chi_{V_A} = \sum_{i=1}^{d} \alpha_i \otimes e_i \, .$$

Lemma

Let $\xi \in K_B \hat{\otimes} K_A$. Then, there is a unique map $\Phi : A(K_A)^+ \to V(K_B)^+$ such that $\xi = (\Phi \otimes \mathrm{id})(\chi_{V(K_A)})$ and $\Phi(\mathbb{1}_{K_A}) \in K_B$.

Effects and measurements

Effects are elementary yes-no-measurements, i.e.,

$$\mathcal{E} := \{\alpha \in A^+ : \mathbb{1} - \alpha \in A^+\}.$$

Definition

Let $(V, V^+, \mathbb{1})$ be a GPT with a state space K. A measurement (or *meter*) on K with $k < \infty$ outcomes is described by a channel $f : K \to S_k$.

We can decompose

$$f(\varrho) = \sum_{i=1}^{k} f_i(\varrho)\delta_i$$

for all $\varrho \in K$, where the f_i are effects summing to $\mathbb{1}$ and the δ_i are the vertices of the simplex. On the other hand, given k effects $f_1, \ldots, f_k$ that satisfy $\sum_{i=1}^{k} f_i = \mathbb{1}$ one can use the above equation to define a channel from K to S_k.

Instruments

We can define instruments as specific types of channels:

Definition

Let $(V_A, V_A^+, \mathbb{1}_{V_A})$ and $(V_B, V_B^+, \mathbb{1}_{V_B})$ be two GPTs with state spaces K_A and K_B, respectively. An instrument from K_A to K_B with k outcomes is described by a channel $\Phi : K_A \to K_B \dot{\otimes} S_k$.

Each k-outcome instrument Φ between K_A and K_B is characterized by a collection of operations $\Phi_1, \ldots, \Phi_k$ such that $\sum_{i=1}^{k} \Phi_i$ is a channel from K_A to K_B which maps

$$\Phi(\varrho) = \sum_{i=1}^{k} \Phi_i(\varrho) \otimes \delta_i$$

for all $\varrho \in K_A$.

GPT of column stochastic matrices

Definition

The *column stochastic GPT* is given by $(CS_{k,g}, CS_{k,g}^{+}, \mathbb{1}_{CS_{k,g}^{1}})$, where $CS_{k,g}$ is the vector space of real $k \times g$ -matrices whose column sums are equal, $CS_{k,g}^{+}$ is the cone of non-negative such matrices and $\mathbb{1}_{CS_{k,g}^{1}} : CS_{k,g} \to \mathbb{R}$ is a functional defined as $\mathbb{1}_{CS_{k,g}^{1}}(M) := \frac{1}{g}\mathrm{Tr}[J_{k,g}M]$ for all $M \in CS_{k,g}$, where $J_{k,g}$ is the $k \times g$ -matrix of all ones. The state space of this GPT is the set of column stochastic $k \times g$ -matrices (non-negative $k \times g$ -matrices whose columns sum to one), which we denote by $CS_{k,g}^{1}$.

Multimeters

Definition

Let $(V, V^+, \mathbb{1})$ be a GPT with a state space K. A multimeter on K with g measurements, each with k outcomes, is described by a channel $f : K \to CS^1_{k,g}$.

Alternatively, we can also represent f as

$$f(\varrho) = \sum_{x=1}^{g} \sum_{a=1}^{k} f_{a|x}(\varrho) E_{a,x} = (f_{\cdot|1}(\varrho), \cdots, f_{\cdot|g}(\varrho))$$

for all $\varrho \in K$. Here, the $E_{a,x}$ are matrix units and the $f_{a|x}$ are effects forming measurements $f_{\cdot|x}$.

Measurement incompatibility as factorizations of multimeters

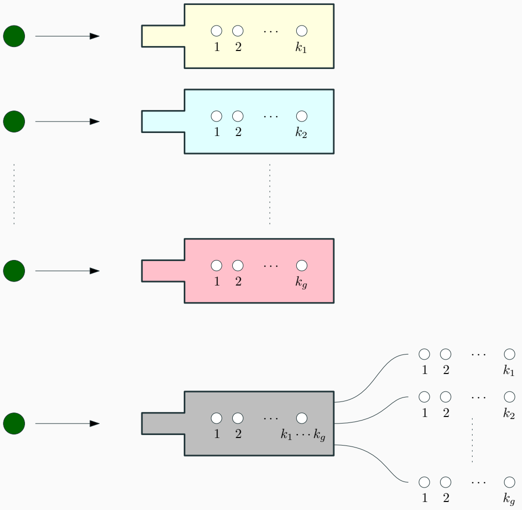
Compatible measurements

Definition

A multimeter $M = \{M_{\cdot|x}\}_{x\in[g]}$ of g measurements with k outcomes on K is compatible if there exists a single l-outcome measurement N on K and some conditional probability distribution $\nu = (\nu_{\cdot|b,x})_{b\in[l],x\in[g]}$ on $[k]$ such that

$$M_{a|x} = \sum_{b=1}^{l} \nu_{a|b,x} N_b$$

for all $a \in [k]$ and $x \in [g]$.



Compatibility as factorization

Theorem

Let $M = \{M_{\cdot|x}\}_{x\in[g]}$ be a multimeter of g measurements with k outcomes on a state space K. Then, the following are equivalent:

1. $(\mathrm{id} \otimes M)(\chi_{V(K)}) \in K^*_\varrho \dot{\otimes} CS^1_{k,g}$ *for some ϱ in the relative interior of K*
2. *M consists of compatible measurements*
3. *There exist a finite number of outcomes Λ, a measurement $N : K \to S_\Lambda$, and a channel $\Phi : S_\Lambda \to CS^1_{k,g}$ such that the following diagram commutes:*

$$\begin{tikzcd} K \arrow[rr, "M"] \arrow[dr, "N"'] & & CS^1_{k,g} \\ & S_\Lambda \arrow[ur, "\Phi"'] & \end{tikzcd}$$

Recovers results from [Jen18]. Here, $K^*_\varrho := \{f \in A(K)^+ : f(\varrho) = 1\}$.

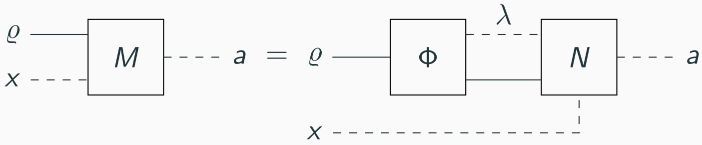
Compression of multimeters

Definition

Let $M = \{M_{\cdot|x}\}_{x\in[g]}$ be a multimeter of g measurements with k outcomes on a state space K_A. It is K_B-*simulable* if there exists a finite number of outcomes Λ, an instrument $\Phi : K_A \to K_B \dot{\otimes} S_\Lambda$ with operations $\Phi_\lambda : V(K_A)^+ \to V(K_B)^+$, and a multimeter $N = \{N_{\cdot|x,\lambda}\}_{x\in[g],\lambda\in[\Lambda]}$ of $g \cdot \Lambda$ measurements with k outcomes on K_B such that

$$M_{a|x} = \sum_{\lambda=1}^{\Lambda} \Phi_\lambda^*(N_{a|x,\lambda})$$

for all $a \in [k]$ and $x \in [g]$. Generalizes an idea from [ISD+22].



Theorem

Let $M = \{M_{\cdot|x}\}_{x\in[g]}$ be a multimeter of g measurements with k outcomes on a state space K_A. Then M is K_B-simulable if and only if there exists a finite number of outcomes Λ, an instrument $\Phi : K_A \to K_B \dot{\otimes} S_\Lambda$ and a multimeter $\tilde{N} : K_B \dot{\otimes} S_\Lambda \to CS^1_{k,g}$ such that the following diagram commutes:

$$\begin{tikzcd} K_A \arrow[rr, "M"] \arrow[dr, "\Phi"'] & & CS^1_{k,g} \\ & K_B \dot{\otimes} S_\Lambda \arrow[ur, "\tilde{N}"'] & \end{tikzcd}$$

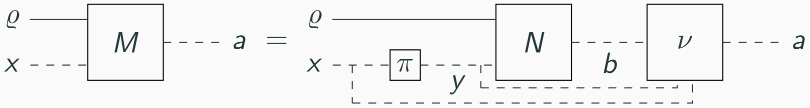
Classical simulation of multimeters

Definition ([FHL18])

Let $M = \{M_{\cdot|x}\}_{x \in g}$ be a multimeter of g measurements with k outcomes on a state space K. We say that M can be *classically simulated* (or is *classically simulable*) with a multimeter $N = \{N_{\cdot|y}\}_{y \in [r]}$ of r measurements with l outcomes on K if there exist conditional probability distributions $\pi = (\pi_{\cdot|x})_{x \in [g]}$ on $[r]$ and $\nu = (\nu_{\cdot|b,x,y})_{b \in [l], x \in [g], y \in [r]}$ on $[k]$ such that

$$M_{a|x} = \sum_{y=1}^{r} \pi_{y|x} \sum_{b=1}^{l} \nu_{a|b,x,y} N_{b|y}$$

for all $a \in [k]$ and $x \in [g]$.



Classical simulation as factorization

Theorem

A multimeter $M = \{M_{\cdot|x}\}_{x\in[g]}$ of g measurements with k outcomes on a state space K can be classically simulated with a multimeter $N = \{N_{\cdot|y}\}_{y\in[r]}$ of r measurements with l outcomes on K if and only if there exists a channel $\Phi : CS^1_{l,r} \to CS^1_{k,g}$ such that the following diagram commutes:

$$\begin{tikzcd} K \arrow[rr, "M"] \arrow[dr, "N"'] & & CS^1_{k,g} \\ & CS^1_{l,r} \arrow[ur, "\Phi"'] & \end{tikzcd}$$

Steering as factorizations of multimeters

Steering in GPTs

Definition

Let $(V(K), V(K)^+, \mathbb{1}_K)$ be a GPT. An *assemblage* is a set $\{q_{a|x}, \varrho_{a|x}\}_{a\in[k],x\in[g]}$ of states $\varrho_{a|x} \in K$ and conditional probability distributions $(q_{a|x})_{a\in[k]}$ for all $x \in [g]$ together with an average state $\bar{\varrho} \in K$ such that

$$\bar{\varrho} = \sum_{a\in[k]} q_{a|x}\varrho_{a|x} \qquad \forall x \in [g] \, .$$

Definition

Let $\{q_{a|x}, \varrho_{a|x}\}_{a\in[k],x\in[g]}$ be an assemblage. Then, the assemblage admits an *LHS model* if there exist $\Lambda \in \mathbb{N}$, a set of states $\{\sigma_\lambda\}_{\lambda\in[\Lambda]}$, a probability distribution $(p_\lambda)_{\lambda\in[\Lambda]}$ and conditional probability distributions $(\nu_{a|\lambda,x})_{a\in[k]}$ for all $\lambda \in [\Lambda]$, $x \in [g]$, such that

$$q_{a|x}\varrho_{a|x} = \sum_{\lambda=1}^{\Lambda} \nu_{a|\lambda,x} p_\lambda \sigma_\lambda \qquad \forall a \in [k], \ \forall x \in [g] \, .$$

Assemblages as multimeters

We can associate to an assemblage a map $\Phi : (CS^*_{k,g})^+ \to V(K)^+$ such that

$$\Phi(\mathbb{1}_{CS^1_{k,g}}) = \bar{\varrho}$$
$$\Phi(m_a^{(x)}) = q_{a|x}\varrho_{a|x} \qquad \forall x \in [g], \forall a \in [k]\,.$$

$m_a^{(x)} = \mathrm{Tr}[E_{a,x}\cdot]$, where $E_{a,x}$ is a matrix unit with a 1 at (a, x). If $\bar{\varrho}$ is in the relative interior of K, the dual map $\Phi^* : K^*_{\bar{\varrho}} \to CS^1_{k,g}$ is a multimeter.

Remark

Warning: Unlike in quantum mechanics, an assemblage cannot always be seen as coming from measuring a bipartite state on one system.

Having an LHS model as factorization

Theorem

Let $\{q_{a|x}, \varrho_{a|x}\}_{a\in[k],x\in[g]}$ *be an assemblage with average state* $\bar{\varrho} \in K$ *and associated map* $\Phi : (CS^*_{k,g})^+ \to V(K)^+$ *such that* $\Phi(\mathbb{1}_{CS^1_{k,g}}) = \bar{\varrho}$*. Then, TFAE:*

1. $(\mathrm{id} \otimes \Phi)(\chi_{CS^*_{k,g}}) \in CS^1_{k,g} \dot{\otimes} K$
2. $\{q_{a|x}, \varrho_{a|x}\}_{a\in[k],x\in[g]}$ *has an LHS model*
3. *There exist a finite number* Λ*, and appropriate maps* Ψ_1*,* Ψ_2 *such that the following diagram commutes:*

$$\begin{tikzcd}
(CS^*_{k,g})^+ \arrow[rr, "\Phi"] \arrow[dr, "\Psi_1"'] & & V(K)^+ \\
& (S^+_\Lambda)^* \arrow[ur, "\Psi_2"'] &
\end{tikzcd}$$

Recovers results from [Jen18].

Classical simulation of assemblages

Definition

Let $\{q_{a|x}, \varrho_{a|x}\}_{a\in[k],x\in[g]}$ be an assemblage with average state $\bar{\varrho} \in K$. We say that the assemblage can be *classically simulated* (or is *classically simulable*) by an assemblage $\{v_{b|y}, \sigma_{b|y}\}_{b\in[l],y\in[r]}$ with $\bar{\sigma} = \bar{\varrho} \in K$ if there exist conditional probability distributions $\pi = (\pi_{\cdot|x})_{x\in[g]}$ on $[r]$ and $\nu = (\nu_{\cdot|b,x,y})_{b\in[l],x\in[g],y\in[r]}$ on $[k]$ such that

$$q_{a|x}\varrho_{a|x} = \sum_{y=1}^{r} \pi_{y|x} \sum_{b=1}^{l} \nu_{a|b,x,y} v_{b|y} \sigma_{b|y}$$

for all $a \in [k]$ and $x \in [g]$.

Motivation: Applying a classically simulable multimeter on half of a bipartite state.

Classical simulation of assemblages as factorization

Theorem

Let $\{q_{a|x}, \varrho_{a|x}\}_{a\in[k],x\in[g]}$ be an assemblage with average state $\bar{\varrho} \in K$ and associated map $\Phi : (CS^*_{k,g})^+ \to V(K)^+$ such that $\Phi(\mathbb{1}_{CS^1_{k,g}}) = \bar{\varrho}$. Then this assemblage is classically simulable by the assemblage $\{v_{b|y}, \sigma_{b|y}\}_{b\in[l],y\in[r]}$ with $\bar{\sigma} = \bar{\varrho} \in K$ if and only if there exist appropriate maps Ψ_1 and Ψ_2 such that the following diagram commutes:

$$\begin{tikzcd}
(CS^*_{k,g})^+ \arrow[rr, "\Phi"] \arrow[dr, "\Psi_1"'] & & V(K)^+ \\
& (CS^*_{l,r})^+ \arrow[ur, "\Psi_2"'] &
\end{tikzcd}$$

If $\bar{\varrho}$ is in the relative interior of K, the assemblage is classically simulable if and only if the multimeter $\Phi^* : K^*_{\bar{\varrho}} \to CS^1_{k,g}$ is classically simulable.

Bell non-locality as factorizations of multimeters

Bell behaviors

- Bipartite Bell behavior:

$$\operatorname{Tr}\left[(M_{a|x} \otimes N_{b|y})\,\varrho\right] = p_{a,b|x,y} \geq 0 \quad \text{with } a \in [k],\, b \in [l],\, x \in [g],\, y \in [r]\,.$$

- Given multimeters $M : K_A \to CS^{+}_{k,g}$, $N : K_B \to CS^{+}_{l,r}$ and $\varrho \in K_A \otimes K_B$, $(M \otimes N)(\varrho)$ is in $CS^{1}_{k,g} \hat{\otimes} CS^{1}_{l,r}$.
- The set of all no-signaling behaviors is in 1-to-1 correspondence with $CS^{1}_{k,g} \hat{\otimes} CS^{1}_{l,r}$.

Definition

A Bell behavior $(p_{a,b|x,y})_{a\in[k],b\in[l],x\in[g],y\in[r]}$ can be described by an LHV model if there is a locally hidden variable $\Lambda \in \mathbb{N}$ and probability distributions q on $[\Lambda]$ and $(v_{\cdot|\lambda,x})$ on $[k]$, $(w_{\cdot|\lambda,y})$ on $[l]$ such that

$$p_{a,b|x,y} = \sum_{\lambda\in[\Lambda]} q_\lambda v_{a|\lambda,x} w_{b|\lambda,y}$$

for all $a \in [k]$, $b \in [l]$, $x \in [g]$, $y \in [r]$.

Bell behaviors as multimeters

Theorem

$\Phi : (CS^+_{k,g})^* \to CS^+_{l,r}$ with $\Phi(\mathbb{1}_{CS^1_{k,g}}) \in CS^1_{l,r}$ if and only if and there exist no-signaling probability distributions $(p_{a,b|x,y})_{a\in[k],b\in[l]}$ such that

$$\langle m_b^{(y)}, \Phi(m_a^{(x)})\rangle = p_{a,b|x,y} \quad \forall a \in [k], \forall b \in [l], \forall x \in [g], \forall y \in [r]$$

If $\Phi(\mathbb{1}_{CS^1_{k,g}}) = q$ is in the relative interior of $CS^1_{k,g}$, then $\Phi : (CS^1_{k,g})^*_q \to CS_{l,r}$ is a multimeter.

Having an LHV model as factorization

Theorem

Let $\Phi : (CS^*_{k,g})^+ \to CS^+_{l,r}$ *with* $\Phi(\mathbb{1}_{CS^1_{k,g}}) \in CS^1_{l,r}$. *Then, the following are equivalent:*

1. $(\Phi \otimes \mathrm{id})(\chi_{CS_{k,g}}) \in CS^1_{l,r} \dot{\otimes} CS^1_{k,g}$.
2. *The associated no-signaling probability distributions* $(p_{a,b|x,y})_{a\in[k],b\in[l]}$ *admit an LHV model.*
3. *There exist a finite number of outcomes* Λ, *and appropriate maps* Ψ_1, Ψ_2 *such that the following diagram commutes:*

$$\begin{tikzcd} (CS^*_{k,g})^+ \arrow[rr, "\Phi"] \arrow[dr, "\Psi_1"'] & & CS^+_{l,r} \\ & S^+_\Lambda \arrow[ur, "\Psi_2"'] & \end{tikzcd}$$

Bell compression as factorization

Theorem

Let $\Phi : (CS^{+}_{k,g})^{*} \to CS^{+}_{l,r}$ with $\Phi(\mathbb{1}_{CS^{1}_{k,g}}) \in CS^{1}_{l,r}$ and let $(V(K), V(K)^{+}, \mathbb{1}_{K})$ be a GPT. Let Ψ and F be appropriate maps. Then, the following diagram commutes

$$\begin{tikzcd}
(CS^{*}_{k,g})^{+} \arrow[rr, "\Phi"] \arrow[dr, "\Psi"'] & & CS^{+}_{l,r} \\
& V(K)^{+} \dot{\otimes} S^{+}_{\Lambda} \arrow[ur, "F"'] &
\end{tikzcd}$$

if and only if the no-signaling probability distributions defining Φ satisfy

$$p_{a,b|x,y} = \sum_{\lambda \in [\Lambda]} w_{\lambda} \langle F_{b|y,\lambda}, q_{a|x,\lambda} \varrho_{a|x,\lambda} \rangle \qquad \forall a \in [k], \forall b \in [l], \forall x \in [g], \forall y \in [r] \,.$$

Here, $(w_{\lambda})_{\lambda \in [\Lambda]}$ is a probability distribution, $(q_{a|x,\lambda})_{a \in [k]}$ are some conditional probability distributions, F is a multimeter, and $\{q_{a|x,\lambda}, \varrho_{a|x,\lambda}\}_{a \in [k], x \in [g]}$ is an assemblage for all $\lambda \in [\Lambda]$.

Bell compression in quantum mechanics

In quantum theory, we can find bipartite states $\varrho_{AB}^{(\lambda)}$ and a multimeter such that $q_{a|x,\lambda}\, \varrho_{a|x,\lambda} = \mathrm{Tr}_A[(E_{a|x,\lambda} \otimes \mathbb{1}_B)\varrho_{AB}^{(\lambda)}]$. Thus, the equation on the previous slide can be written as

$$p_{a,b|x,y} = \sum_{\lambda=1}^{\Lambda} w_\lambda \, \mathrm{Tr}\left[(E_{a|x,\lambda} \otimes F_{b|y,\lambda})\varrho_{AB}^{(\lambda)}\right] \qquad \forall a \in [k], \forall b \in [l], \forall x \in [g], \forall y \in [r]\,.$$

In GPTs, this might not be possible, depending on the bipartite theory.

Extendability of tensors

Extendability of tensors

Entanglement theory: ϱ_{AB} is separable if and only if for all $n \in \mathbb{N}$ there exists a state $\sigma_{AB_1B_2\ldots B_n}$ that extends ϱ_{AB} , i.e.,

1. $\mathrm{Tr}_{B_2\ldots B_n}[\sigma_{AB_1B_2\ldots B_n}] = \varrho_{AB}$
2. $\pi_{B_1B_2\ldots B_n}(\sigma_{AB_1B_2\ldots B_n}) = \sigma_{AB_1B_2\ldots B_n}$, where π permutes $B_1, \ldots, B_n$.

Thus, separability can be checked using the DPS hierarchy [DPS04].

To extend this idea to GPTs, we define a reduction map of n-th order:

$$\gamma_n^{\Phi} : V_B^{\otimes n} \to V_B, \quad \gamma_n^{\Phi} = \frac{1}{n}\sum_{i=1}^{n} \Phi^{\otimes i-1} \otimes \mathrm{id} \otimes \Phi^{\otimes n-i},$$

Definition

Let Φ be a linear form in the interior of $(V_B^+)^*$. A tensor $\xi \in V_A^+ \hat{\otimes} V_B^+$ is n-extendable with respect to Φ if there exists $\xi^{(n)} \in V_A^+ \hat{\otimes} (V_B^+)^{\otimes n}$ such that $(id \otimes \gamma_n^{\Phi})(\xi^{(n)}) = \xi$.

Extendability of multimeters

Definition

A multimeter $M = \{M_{\cdot|x}\}_{x\in[g]}$ is said to be n-wise compatible with $n \leq g$ if for all $\{x_1, \dots, x_n\} \subset [g]$ the measurements $\{M_{\cdot|x_i}\}_{i\in[n]}$ are compatible.

Theorem

Let $M : K \to CS^1_{k,g}$ be a multimeter and $\xi_M \in A(K)^+ \hat{\otimes} CS^+_{k,g}$ be the corresponding tensor. Then M is n-extendable with $\Phi = \mathbb{1}_{CS^1_{k,g}}$ if and only if it is n-wise compatible such that for all measurements $\{x_1, \dots, x_n\} \subset [g]$ there exists joint measurements $G^{(x_1,\dots,x_n)} = \{G^{(x_1,\dots,x_n)}_{a_1,\dots,a_n}\}^k_{a_1,\dots,a_n=1}$ which satisfy the following no-signaling constraints

$$\sum_{a_i=1}^{k} G^{(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots x_n)}_{a_1,\dots,a_n} = \sum_{a_i=1}^{k} G^{(x_1,\dots,x_{i-1},\tilde{x}_i,x_{i+1},\dots x_n)}_{a_1,\dots,a_n}$$

for all $a_j \in [k]$, $x_j \in [g]$, $j \in [n] \setminus \{i\}$ and $x_i, \tilde{x}_i \in [g]$ for all $i \in [n]$. In particular, M is compatible if and only if ξ_M is g-extendable.

Extendability of no-signaling behaviors

Definition

Let $g, r, k, l \in \mathbb{N}$ and $m \in [g]$ and $n \in [r]$. A no-signaling probability distribution $p = (p_{\cdot,\cdot|x,y})_{x\in[g],y\in[r]}$ on $[k] \times [l]$ has an (m, n)-LHV model if for all $(x_1, \ldots, x_m) \subseteq [g]$ and $(y_1, \ldots, y_n) \subseteq [r]$ the restricted no-signaling probability distribution $(p_{\cdot,\cdot|x,y})_{x\in\{x_1,\ldots,x_m\},y\in\{y_1,\ldots,y_n\}}$ on $[k] \times [l]$ has an LHV model.

Note that $p = (p_{\cdot,\cdot|x,y})_{x\in[g],y\in[r]}$ has an LHV model if and only if it has an (g, r)-LHV model.

Theorem

Let $P \in CS^1_{k,g} \hat{\otimes} CS^1_{k,g}$ with the corresponding no-signaling probability distribution $p = (p_{\cdot,\cdot|x,y})_{x,y\in[g]}$ on $[k] \times [k]$ and let $n \in [g]$. Then P is n-extendable if and only if p has an (g, n)-LHV model such that the corresponding (g, n)-LHV decompositions satisfy some additional no-signaling constraints. If $n = g$, then $P \in CS^1_{k,g} \dot{\otimes} CS^1_{k,g}$.

Summary

- We have presented the formalism of general probabilistic theories (GPTs).
- In particular, we have seen how to write multimeters, i.e., collections of measurements, as channels from the state space to column stochastic matrices.
- We have used it to define measurement incompatibility for GPTs.
- We have expressed compatibility of measurements and different notions of simulations as factorizations of multimeters.
- We have then seen that the same ideas carry over to assemblages and no-signaling behaviors.
- We have seen that multimeters are n-extendable if and only if the measurements are n-wise compatible and the joint measurements satisfy no-signaling constraints.
- Likewise, we have seen that no-signaling behaviors are n-extendable if and only if the they have a (g, n)-LHV model, satisfying some no-signaling constraints.

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