

Computing Noise Robustness of Incompatible Quantum Measurements

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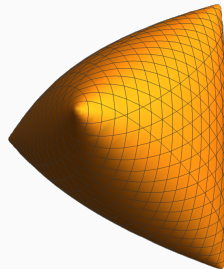
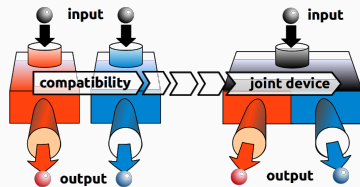
Talk outline

Incompatibility in QM

Free spectrahedra

Connecting the two

Computing the noise robustness of incompatibility



Incompatibility in QM

Quantum states and measurements

- Motivation: Classical state $\rightsquigarrow$ **probability distributions**: $p \in \mathbb{R}^d$, $p \geq 0$, $\sum_i p_i = 1$
- Quantum states $\rightsquigarrow$ **density matrices**: $\rho \in \mathcal{M}_d(\mathbb{C})$, $\rho \geq 0$, $\text{Tr } \rho = 1$
- Measurement outcomes are labeled $\{1, \dots, k\}$, need to be assigned probabilities
- **Measurements**: Tuples of matrices $(E_1, \dots, E_k)$ such that $(\text{Tr}[E_1\rho], \dots, \text{Tr}[E_k\rho])$ is a probability distribution for all states ρ
 - $\text{Tr}[E_i\rho] \in \mathbb{R} \rightsquigarrow E_i = E_i^*$
 - $\text{Tr}[E_i\rho] \geq 0 \rightsquigarrow E_i \geq 0$
 - $\sum_i \text{Tr}[E_i\rho] = 1 \rightsquigarrow \sum_i E_i = I_d$
- Tuples of PSD matrices summing to identity are called positive operator-valued measures (**POVMs**)
- We call $0 \leq E \leq I$ **quantum effects**

Quantum measurements: Compatibility

- Quantum measurements $\rightsquigarrow$ give the probabilities of the classical outcomes when a quantum state enters a measurement apparatus. Mathematically, measurements are modeled by POVMs

Definition

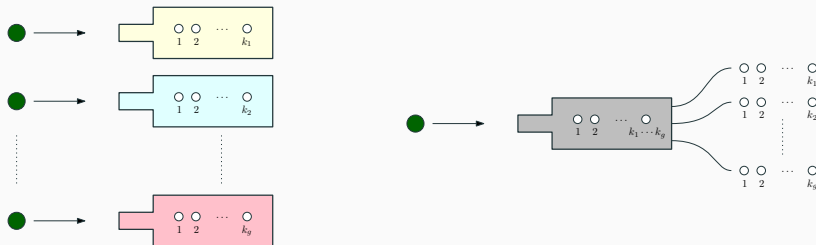
Two POVMs, $A = (A_1, \dots, A_k)$ and $B = (B_1, \dots, B_l)$, are called **compatible** if there exists a third POVM $C = (C_{ij})_{i \in [k], j \in [l]}$ such that

$$\forall i \in [k], \quad A_i = \sum_{j=1}^l C_{ij} \quad \text{and} \quad \forall j \in [l], \quad B_j = \sum_{i=1}^k C_{ij}$$

The definition generalizes to g -tuples of POVMs $A^{(1)}, \dots, A^{(g)}$, having respectively $k_1, \dots, k_g$ outcomes, where the **joint** POVM C has outcome set $[k_1] \times \dots \times [k_g]$.

- Other way to say that: **jointly measurable**

What does it mean?



- Compatible measurements can be simulated by a single joint measurement, by **classically post-processing** its outputs
- Examples:
 1. **Trivial** POVMs $A = (p_i I_d)$ and $B = (q_j I_d)$ are compatible
 2. **Commuting** POVMs $[A_i, B_j] = 0$ are compatible
 3. If the POVM A is **projective**, then A and B are compatible iff they commute

Noisy POVMs

- POVMs can be made compatible by adding **noise**, i.e. mixing in trivial POVMs
- Example: dichotomic POVMs and white noise, $s \in [0, 1]$

$$(E, I - E) \mapsto s(E, I - E) + (1 - s) \left(\frac{I}{2}, \frac{I}{2} \right) \quad \text{or} \quad E \mapsto sE + (1 - s) \frac{I}{2}$$

- Taking $s = 1/2$ suffices to render any pair of dichotomic POVMs compatible $\rightsquigarrow$
define $C_{ij} := (E_i + F_j)/4$
- For most of the talk, we focus on dichotomic (YES/NO) POVMs

Definition

The **incompatibility degree** for g measurements on $\mathbb{C}^d$ is the number

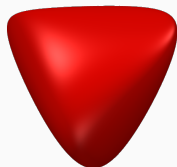
$$\gamma(g, d) := \max\{s \in [0, 1] : \text{for all quantum effects } E_1, \dots, E_g \in \mathcal{M}_d(\mathbb{C}), \\ \text{the noisy versions } sE_i + (1 - s)I_d/2 \text{ are compatible}\}$$

Free spectrahedra

Free spectrahedra

- A **spectrahedron** is given by PSD constraints: for $A = (A_1, \dots, A_g) \in (\mathcal{M}_d(\mathbb{C})^{\text{sa}})^g$

$$\mathcal{D}_A(1) := \left\{ x \in \mathbb{R}^g : \sum_{i=1}^g x_i A_i \leq I_d \right\}$$



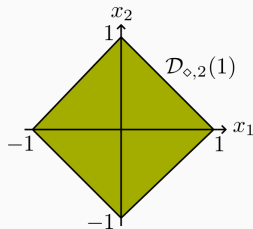
- $\mathcal{D}_{(\sigma_X, \sigma_Y, \sigma_Z)}(1) = \{(x, y, z) \in \mathbb{R}^3 : x\sigma_X + y\sigma_Y + z\sigma_Z \leq I_2\} = \text{Bloch ball}$
- A **free spectrahedron** is the matricization of a spectrahedron

$$\mathcal{D}_A := \bigsqcup_{n=1}^{\infty} \mathcal{D}_A(n) \quad \text{with} \quad \mathcal{D}_A(n) := \left\{ X \in (\mathcal{M}_n(\mathbb{C})^{\text{sa}})^g : \sum_{i=1}^g X_i \otimes A_i \leq I_{nd} \right\}$$

Example: the matrix diamond

The **matrix diamond** is the free spectrahedron defined by

$$\mathcal{D}_{\diamond,g} := \bigsqcup_{n=1}^{\infty} \left\{ X \in (\mathcal{M}_n(\mathbb{C})^{\text{sa}})^g : \sum_{i=1}^g \epsilon_i X_i \leq I_n, \quad \forall \epsilon \in \{\pm 1\}^g \right\}$$



- At level one, $\mathcal{D}_{\diamond,g}(1)$ is the unit ball of the ℓ^1 norm on $\mathbb{R}^g$
- As a free spectrahedron, it is defined by $2^g \times 2^g$ diagonal matrices $\mathcal{D}_{\diamond,g} = \mathcal{D}_{L_1, \dots, L_g}$, with $L_i = I_2 \otimes \dots \otimes I_2 \otimes \text{diag}(1, -1) \otimes I_2 \otimes \dots \otimes I_2$

Spectrahedral inclusion

- Consider two free spectrahedra defined by $(A_1, \dots, A_g)$ and $(B_1, \dots, B_g)$
- We write $\mathcal{D}_A \subseteq \mathcal{D}_B$ if, for all $n \geq 1$, $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$
- Clearly, $\mathcal{D}_A \subseteq \mathcal{D}_B \implies \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$. For the converse implication to hold, one may need to shrink $\mathcal{D}_A$

Definition

For a free spectrahedron $\mathcal{D}_A$, we define its **inclusion constant** as

$$\delta_A(g, d) := \max\{s \in [0, 1] : \text{for all } g\text{-tuples } B_1, \dots, B_g \in \mathcal{M}_d(\mathbb{C})^{\text{sa}}, \\ \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1) \implies s \cdot \mathcal{D}_A \subseteq \mathcal{D}_B\}$$

- We shall be concerned with the inclusion constant for the **matrix diamond**, which we denote by $\delta(g, d)$

Connecting the two

Compatibility in QM $\iff$ matrix diamond inclusion

To a g -tuple $E \in (\mathcal{M}_d(\mathbb{C})^{\text{sa}})^g$, we associate:

$$\mathcal{D}_{2E-I} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n(\mathbb{C})^{\text{sa}})^g : \sum_{i=1}^g X_i \otimes (2E_i - I_d) \leq I_{nd}\}$$

Theorem

Let $E \in (\mathcal{M}_d(\mathbb{C})^{\text{sa}})^g$ be g -tuple of selfadjoint matrices. Then:

- The matrices E are *quantum effects* $\iff \mathcal{D}_{\diamond,g}(1) \subseteq \mathcal{D}_{2E-I}(1)$
- The matrices E are *compatible* quantum effects $\iff \mathcal{D}_{\diamond,g} \subseteq \mathcal{D}_{2E-I}$

At the intermediate levels $1 \leq n \leq d$, $\mathcal{D}_{\diamond,g}(n) \subseteq \mathcal{D}_{2E-I}(n)$ iff for all isometries $V : \mathbb{C}^n \rightarrow \mathbb{C}^d$, the compressed effects $V^* E_i V$ are compatible.

Moreover, the incompatibility degree is equal to the inclusion constant of the matrix diamond: $\forall g, d, \gamma(g, d) = \delta(g, d)$.

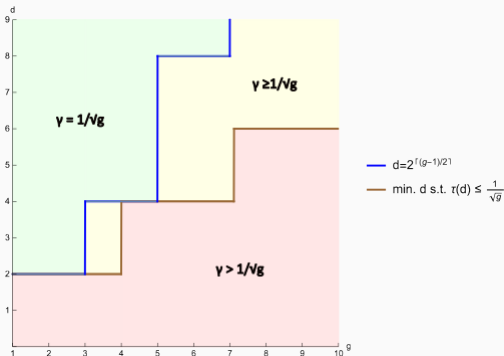
Many things are known about the matrix diamond:

- For all g, d , $\tau(d) \leq \delta(g, d)$, $\tau(d) \approx \sqrt{\frac{2}{\pi d}}$ asymptotically (AB and Nechita, 2022)
- For all g, d , $\frac{1}{\sqrt{g}} \leq \delta(g, d)$ (Passer *et al.*, 2018)

Theorem (Passer *et al.*, 2018)

For all g and $d \geq 2^{\lceil (g-1)/2 \rceil}$, $\gamma(g, d) = \delta(g, d) = \frac{1}{\sqrt{g}}$

Phase diagram



- Connection to free spectrahedra also holds for arbitrary outcomes
- Instead of matrix diamond, consider its generalization, the [matrix jewel](#)
- The [smallest unknown case](#) is $d = 2, g = 4$ (4 qubit measurements)

Computing the noise robustness of incompatibility

4 qubit measurements: Reminder

- Connection between compatibility and free spectrahedra: The matrices E are **compatible** quantum effects $\iff \mathcal{D}_{\diamond, g} \subseteq \mathcal{D}_{2E-I}$
- $B_i = 2E_i - I$ for some $I \geq E_i \geq 0$ $\iff -I \leq B_i \leq I$
- $X \in \mathcal{D}_{\diamond, 4}$ $\iff \sum_{i=1}^4 \epsilon_i X_i \leq I_2 \quad \forall \epsilon \in \{\pm 1\}^4$
- $X \in \mathcal{D}_{2E-I}$ $\iff \sum_{i=1}^4 B_i \otimes X_i \leq I$

Goal: Maximize

$$\lambda_{\max} \left(\sum_{i=1}^4 B_i \otimes X_i \right)$$

over all allowed X, B as above.

Value: $1/\delta(4, 2) = 1/\gamma(4, 2)$

4 qubit measurements: Optimization problem

$$\begin{aligned} & \text{maximize} && \lambda_{\max} \left(\sum_{i=1}^4 B_i \otimes X_i \right) \\ & \text{subject to} && B_1 \leq I_2, \quad -B_1 \leq I_2, \\ & && B_2 \leq I_2, \quad -B_2 \leq I_2, \\ & && B_3 \leq I_2, \quad -B_3 \leq I_2, \\ & && B_4 \leq I_2, \quad -B_4 \leq I_2, \\ & && \sum_{i=1}^4 \epsilon_i X_i \leq I_2, \quad \forall \epsilon \in \{\pm 1\}^4 \\ & && B_i, X_i \in \mathcal{M}_2(\mathbb{C})^{\text{sa}} \quad \forall i \in \{1, 2, 3, 4\} \end{aligned}$$

Problem: Need to keep the dimension **fixed**, otherwise the result is $1/2$ ($1/\sqrt{g}$, $g = 4$)

Bloch sphere expansion

- Use Pauli matrices to write

$$B_i = b_0^{(i)} I_2 + b_1^{(i)} \sigma_X + b_2^{(i)} \sigma_Y + b_3^{(i)} \sigma_Z$$

$$X_i = x_0^{(i)} I_2 + x_1^{(i)} \sigma_X + x_2^{(i)} \sigma_Y + x_3^{(i)} \sigma_Z.$$

- Can convert maximal eigenvalue into a trace using the fact that we can write any state as

$$|\psi\rangle = (I \otimes X_\psi) |\Omega\rangle$$

with $|\Omega\rangle$ being the maximally entangled state

- We obtain a polynomial optimization problem in commuting variables
- Solve it with Lasserre-Parillo hierarchy

Commutative polynomial optimization problem

$$\begin{aligned} & \text{maximize} && \sum_i \sum_j b_j^{(i)} x_j^{(i)} \\ & \text{subject to} && 1 - (b_1^{(0)})^2 - (b_2^{(0)})^2 - (b_3^{(0)})^2 \geq 0, \\ & && (1 - b_0^{(i)})^2 - (b_1^{(0)} - b_1^{(i)})^2 - (b_2^{(0)} - b_2^{(i)})^2 - (b_3^{(0)} - b_3^{(i)})^2 \geq 0, \quad \forall i \in [4] \\ & && (1 + b_0^{(i)})^2 - (b_1^{(0)} + b_1^{(i)})^2 - (b_2^{(0)} + b_2^{(i)})^2 - (b_3^{(0)} + b_3^{(i)})^2 \geq 0, \quad \forall i \in [4] \\ & && 1 \geq b_0^{(i)} \geq -1, \quad \forall i \in \{1, 2, 3, 4\} \\ & && (1 - \sum_i \epsilon_i x_0^{(i)})^2 \geq \sum_j (\sum_i \epsilon_i x_j^{(i)})^2, \quad \forall \epsilon \in \{\pm 1\}^4 \\ & && 1 \geq \sum_i \epsilon_i x_0^{(i)}, \quad \forall \epsilon \in \{\pm 1\}^4 \\ & && b_j^{(i)}, x_j^{(i)} \in \mathbb{R} \quad \forall i, j \end{aligned}$$

Conjectured noise robustness of 4 qubit measurements

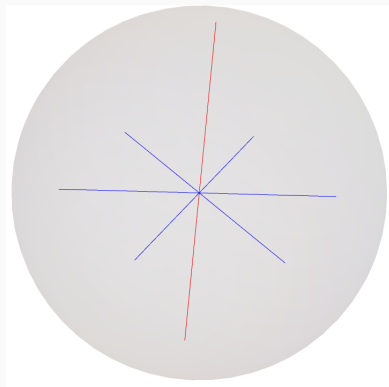
Known constraints:

- $\gamma(g, d) \geq 1/\sqrt{g} \implies \gamma(4, 2) \geq 1/2$
- $\gamma(g, d) \geq \gamma(g+1, d)$ and $\gamma(3, 2) = 1/\sqrt{3} \implies \gamma(4, 2) \leq 1/\sqrt{3} \approx 0.58$

Conjecture: $\gamma(4, 2) = 2/\sqrt{13} \approx 0.55$

- Theory of extreme points for free spectrahedra: $\gamma(4, 2) \leq 2/\sqrt{13}$
- Polynomial optimization: Almost matching lower bounds (up to $\mathcal{O}(10^{-4})$)

Consistent with numerical results by Bavaresco *et al.*, 2017 (not PO)



More outcomes

- The connection between compatibility and free spectrahedra also holds for more outcomes
- Need to substitute the matrix diamond by the matrix jewel
- Can for example consider one measurement with 2 and one measurement with k outcomes
- Not much is known for this problem
- Can write down a polynomial optimization problem again to compute the noise robustness

Polynomial optimization problem for 2 + 3 outcomes

$$\begin{aligned} & \text{maximize} && \lambda_{\max} \left(\sum_{i=1}^3 B_i \otimes X_i \right) \\ & \text{subject to} && B_1 \leq I_m, \quad -B_1 \leq I_m, \\ & && -\frac{3}{2}B_2 \leq I_m, \quad -\frac{3}{2}B_3 \leq I_m, \\ & && \frac{3}{2}B_2 + \frac{3}{2}B_3 \leq I_m, \\ & && \pm X_1 + \frac{4}{3}X_2 - \frac{2}{3}X_3 \leq I_m, \\ & && \pm X_1 - \frac{2}{3}X_2 + \frac{4}{3}X_3 \leq I_m, \\ & && \pm X_1 - \frac{2}{3}X_2 - \frac{2}{3}X_3 \leq I_m, \\ & && B_i, X_i \in \mathcal{M}_m(\mathbb{C})^{\text{sa}} \quad \forall i \in \{1, 2, 3\} \end{aligned}$$

If we don't fix the dimension m , we can solve this problem using the NPA hierarchy

Conjectured noise robustness of $2 + k$ outcomes measurements

- For $m \geq 2$, we know that $\gamma((2, 2), m) = \frac{1}{\sqrt{2}}$ and that $1/2 \leq \gamma((2, 3), m) \leq 1/\sqrt{2}$

Conjecture: $\gamma((2, k), m) = \frac{1}{2} \left(1 + \frac{1}{1+\sqrt{k}}\right)$ for all $m \geq 2, k \geq 2$

- Theory of extreme points for free spectrahedra: $\gamma((2, k), m) \leq \frac{1}{2} \left(1 + \frac{1}{1+\sqrt{k}}\right)$
- Corresponding measurements: $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ and $\{|+\rangle\langle +|, |-\rangle\langle -|, 0, \dots, 0\}$
- For $k \in \{2, 3, 4, 5\}$: NPA hierarchy almost matching lower bounds up to high precision

Analytic results from extreme points of free spectrahedra

If we restrict to **real** matrices in the optimization problem and $m = 2$, we can actually **prove** that $\gamma((2, k), 2) = \frac{1}{2} \left(1 + \frac{1}{1+\sqrt{k}} \right)$ for all $k \geq 2$

What keeps us from generalizing to all m ?

- A linear map Φ is n -positive iff $\Phi \otimes \text{id}_n$ is positive
- If $\Phi : S \rightarrow \mathcal{M}_{d_B}(\mathbb{C})$, where $S \subseteq \mathcal{M}_{d_A}(\mathbb{C})$ is an operator system, then Φ is d_B positive iff Φ is CP. However, if we only know d_A , similar implications no longer hold

Why only real matrices?

- Connected to the extreme points of $(\mathcal{D}_{\Delta_k} \times \mathcal{D}_{\Delta_2})(2)$, where $\mathcal{D}_{\Delta_k}$ is a free simplex
- **Real**: If (X, Y) is matrix extreme, then X and Y are free extreme
- This is good, because there are very few free extreme points
- **Complex**: If X is Euclidean (not free) extreme and Y is free extreme with non-zero imaginary part, then (X, Y) is matrix extreme

Open questions

- Can we simplify the optimization problems further using knowledge about extreme points of free spectrahedra (in particular for 4 dichotomic qubit measurements)?
- Is there a way to extract exact certificates from the polynomial optimization?
- Can we solve the polynomial optimization problem more efficiently (allowing to tackle larger problems)?

References

Inclusion constants:

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Compatibility and free spectrahedra:

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