Strong decay of correlations for Gibbs states in any dimension

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Motivation

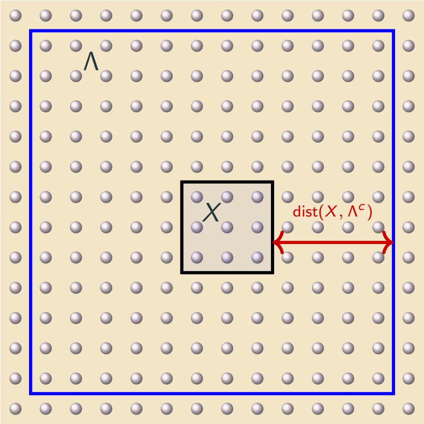
First question: How difficult is it to describe quantum many-body systems in thermal equilibrium?

- For local Hamiltonians and high enough temperatures, the answer should be: not too difficult
- It is known that such systems can efficiently be simulated classically using, e.g., tensor networks
- Reason: Fast decay of correlations
- There are many different ways to measure correlations

Second question: Which correlation measures are equivalent?

Preliminaries

Formalism: Quantum spin systems



- Graph: $\mathbb{Z}^g$ with a distance dist
- At each vertex x Hilbert space $\mathcal{H}_x \simeq \mathbb{C}^d$
- On finite subset X: $\mathcal{H}_X = \bigotimes_{x\in X} \mathcal{H}_x \simeq (\mathbb{C}^d)^{\otimes|X|}$
- Observables: $\mathfrak{A}_X = \mathcal{B}(\mathcal{H}_X) \simeq \mathcal{M}_d(\mathbb{C})^{\otimes|X|}$
- Identify for $X \subset \Lambda$: $\mathfrak{A}_X \hookrightarrow \mathfrak{A}_\Lambda$, $Q \mapsto Q \otimes \mathbb{1}_{\Lambda\setminus X}$

Formalism: Gibbs states of local Hamiltonians

- Interactions Φ: For each finite X, $\Phi_X \in \mathfrak{A}_X$ and $\Phi_X^* = \Phi_X$
- Finite range: $\Phi_X = 0$ if $\mathrm{diam}(X) > r$, $\|\Phi_X\|_\infty \leq J$ for all finite X
- Short range or exponentially decaying: $\|\Phi\|_{\lambda,\mu} \leq \infty$, where

$$\|\Phi\|_{\lambda,\mu} = \sup_{x\in\mathbb{Z}^g} \sum_{X\ni x} \|\Phi_X\|_\infty e^{\lambda|X|+\mu\,\mathrm{diam}(X)}$$

- Hamiltonian on finite subset Λ: $H_\Lambda = \sum_{X\subset\Lambda} \Phi_X$
- Gibbs state on Λ at inverse temperature β:

$$\rho_\beta^\Lambda = \frac{e^{-\beta H_\Lambda}}{\mathrm{Tr}_\Lambda[e^{-\beta H_\Lambda}]}$$

- Marginals of the Gibbs state: $\rho_{\beta,X}^\Lambda = \mathrm{tr}_{\Lambda\setminus X}[\rho_\beta^\Lambda]$

Covariance

First measure of correlations: operator of covariance correlation

$$\mathrm{Cov}_\rho(A, C) = \sup_{O_A, O_C} \left|\mathrm{Tr}[O_A \otimes O_C(\rho_{AC} - \rho_A \otimes \rho_C)]\right| ,$$

where supremum runs over $O_A \in \mathfrak{A}_A$, $O_C \in \mathfrak{A}_C$ with norm at most 1

Definition (Exponential uniform decay of covariance)

A system exhibits exponential uniform decay of covariance at inverse temperature β if there exist universal constants $K, \alpha \geq 0$ such that for all finite $\Lambda \subset \mathbb{Z}^g$, $\rho := \rho_\beta^\Lambda$ the Gibbs state for the Hamiltonian on the region Λ, and $A, C \subset \Lambda$ with $A \cap C \neq \emptyset$,

$$\mathrm{Cov}_\rho(A, C) \leq K\, f(A, C)\, \mathrm{e}^{-\alpha\, \mathrm{dist}(A,C)} .$$

Here, $f(A, C)$ is a function of A and C, for example $f(A, C) = 1$, $f(A, C) = |\partial A||\partial C|$, or $f(A, C) = |A||C|$.

Mutual information

Second measure of correlations: mutual information $I_\rho(A:C) = D(\rho_{AC} \| \rho_A \otimes \rho_C)$, with

$$D(\rho\|\sigma) = \begin{cases} \mathrm{Tr}[\rho \log \rho - \rho \log \sigma] & \mathrm{supp}(\rho) \subset \mathrm{supp}(\sigma) \\ \infty & \text{else} \end{cases} .$$

Definition (Exponential uniform decay of mutual information)

A system exhibits exponential uniform decay of mutual information at inverse temperature β if there exist universal constants $K, \alpha \geq 0$ such that for all finite $\Lambda \subset \mathbb{Z}^g$, $\rho := \rho_\beta^\Lambda$ the Gibbs state for the Hamiltonian on the region Λ, and $A, C \subset \Lambda$ with $A \cap C \neq \emptyset$,

$$I_\rho(A:C) \leq K\, f(A,C)\, \mathrm{e}^{-\alpha\, \mathrm{dist}(A,C)} .$$

$I_\rho(A:C) \geq \frac{1}{2} \mathrm{Corr}_\rho(A:C)^2$ due to Pinsker's inequality and the inequality can be strict (data hiding)

Mixing condition

Third measure of correlations: mixing condition. Considers the quantity

$$\|\rho_{AC}\,\rho_A^{-1}\otimes\rho_C^{-1} - \mathbb{1}_{AC}\|_\infty \geq I_\rho(A:C)$$

Definition (Uniform mixing condition)

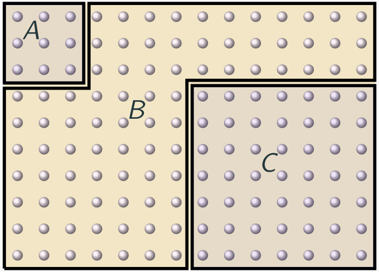
A uniform mixing condition holds at inverse temperature β if there exist universal constants $K, \alpha \geq 0$ such that for all finite $\Lambda \subset \mathbb{Z}^g$, $\rho := \rho_\beta^\Lambda$ the Gibbs state for the Hamiltonian on the region Λ, and $A, C \subset \Lambda$ with $A \cap C \neq \emptyset$,

$$\|\rho_{AC}\,\rho_A^{-1}\otimes\rho_C^{-1} - \mathbb{1}_{AC}\|_\infty \leq K\, f(A,C)\ \mathrm{e}^{-\alpha\,\mathrm{dist}(A,C)}\ .$$

The name comes from the study of modified logarithmic Sobolev inequalities. These can be used to show that systems are rapidly mixing and therefore unsuitable as self-correcting quantum memories.

Local indistinguishability

Fourth measure of correlations: local indistinguishability



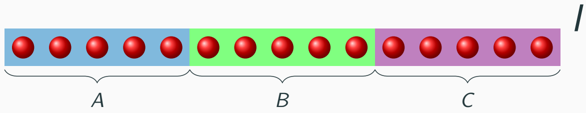
Definition (Local indistinguishability)

Uniform local indistinguishability holds if there exist universal constants $K, \alpha \geq 0$ such that for all finite $\Lambda \subset \mathbb{Z}^g$, split as $\Lambda = ABC$ with B shielding A from C, and for all local operators O_A on A,

$$\left| \mathrm{Tr}_{ABC}[\rho^{\Lambda} O_A] - \mathrm{Tr}_{AB}[\rho^{AB} O_A] \right| \leq \|O_A\|_\infty f(A,C) \, K \, \mathrm{e}^{-\alpha \, \mathrm{dist}(A,C)} \, .$$

Decay of correlations in one dimension

Mixing condition in one dimension



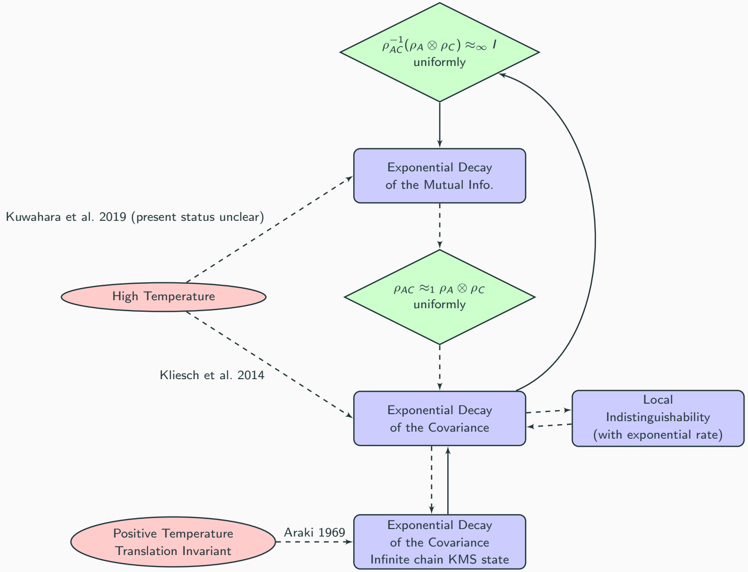
Theorem

We consider again a quantum spin system on $\mathbb{Z}$ with finite-range, translation-invariant interactions. In this setting, there exists a positive function $\ell \mapsto \delta_1(\ell)$, depending on the local interactions and exhibiting exponential decay, such that for every finite interval $I \subset \mathbb{Z}$ split into three subintervals $I = ABC$, where B shields A from C, the Gibbs state $\rho = e^{-H_I} / \operatorname{Tr}\left(e^{-H_I}\right)$ satisfies

$$I_\rho(A:C) \;\leq\; \left\| \rho_A^{-1} \otimes \rho_C^{-1} \rho_{AC} - \mathbb{1} \right\|_\infty \;\leq\; \delta_1(|B|)\,.$$

In other words, the mixing conditions holds at any finite non-zero temperature.

Equivalence between different measures of decay in one dimension



Araki's expansionals

Lemma (Araki 1969)

Let Φ be a local interaction with range r and strength J. For a finite interval $I = XY \subset \mathbb{Z}$ split into two subintervals X and Y, let us write

$$E_{X,Y} = e^{-H_{XY}} e^{H_X + H_Y} .$$

Then, there is an absolute constant $\mathcal{G} > 1$ depending only on J and r such that:

(i) $\|E_{X,Y}\|, \|E_{X,Y}^{-1}\| \leq \mathcal{G}$

(ii) If we add two intervals $\widetilde{X}$ and $\widetilde{Y}$ adjacent to X and Y, respectively, so that we get a larger interval $\widetilde{J} := \widetilde{X}XY\widetilde{Y}$, then

$$\left\| E_{X,Y}^{-1} - E_{\widetilde{X}X,Y\widetilde{Y}}^{-1} \right\|, \left\| E_{X,Y} - E_{\widetilde{X}X,Y\widetilde{Y}} \right\| \leq \frac{\mathcal{G}^{\ell}}{(\lfloor \ell / r \rfloor + 1)!} .$$

for any $\ell \in \mathbb{N}$ such that $\ell \leq |X|, |Y|$.

Follows from Araki's complex time generalization of Lieb-Robinson bounds.

Decay of correlations in higher dimensions

Araki's expansionals

Reminder: $E_{X,Y}(s) = \mathrm{e}^{-sH_{XY}}\, \mathrm{e}^{sH_X + sH_Y} \qquad$ for every $s \in \mathbb{C}$

Lemma

Let A, B, C be disjoint finite sets and let Φ be a local interaction satisfying for some $\lambda, \mu > 0$

$$\|\Phi\| = \|\Phi\|_{\lambda,\mu} := \sup_{x \in V} \sum_{X \ni x} \|\Phi_X\|_\infty \mathrm{e}^{\lambda|X| + \mu\, \mathrm{diam}(X)} < \infty\,,$$

Then, for every real number β with $|\beta| < \frac{\lambda}{2\|\Phi\|}$ we have

$$\|E_{A,B}(\beta)\|_\infty \leq \exp\{|\beta| \min\{|\partial A|, |\partial B|\} K\}\,,$$

and

$$\|E_{A,BC}(\beta) - E_{A,B}(\beta)\|_\infty \leq \exp[|\beta| K |\partial A|] K' |\partial A| \exp\{-\frac{\mu}{2} \mathrm{dist}(A, C)\}\,.$$

Quantum belief propagation

- Was introduced by Hastings in 2007
- Consider a finite-range Hamiltonian H with perturbation W. We can write (using Dyson series):
$$e^{-\beta(H+W)} = \eta(W)e^{-\beta H}\eta(W)^*,$$
with $\|\eta(W)\| \leq e^{\beta\|W\|/2}$.
- Can find a local approximation $\eta_\ell(W)$ supported within distance ℓ of supp W and with $\|\eta_\ell(W)\| \leq e^{\beta\|W\|/2}$. Approximation (using Lieb-Robinson bounds):
$$\|\eta(W) - \eta_\ell(W)\| \leq e^{c_1\|W\|}e^{-c_2\ell}.$$
- Similar statements hold for short-range instead of finite-range interactions (Capel et al. 2023).

Effective Hamiltonian - strong form

We can always write $\widetilde{H}_\Lambda^{L,\beta} := -\frac{1}{\beta}\log \mathbb{E}_L[e^{-\beta H_\Lambda}]$. We would hope that if H_Λ is local, so is the effective Hamiltonian $\widetilde{H}_\Lambda^{L,\beta}$ at high enough temperature.

Definition

Let us say that the above quantum spin system has (strong) local effective Hamiltonians at (inverse) temperature $\beta > 0$ if it satisfies the following property: for every $L \subset V$, there exists a local interaction $\widetilde{\Phi}^{L,\beta}$ on V satisfying

(i) $\widetilde{\Phi}_X^{L,\beta}$ is supported in $X \cap L$ for every finite $X \subset V$.

(ii) If $L' \subset V$, then $\widetilde{\Phi}_X^{L,\beta} = \widetilde{\Phi}_X^{L',\beta}$ for all finite $X \subset X$ such that $X \cap L' = X \cap L$.

(iii) For every finite subset $\Lambda \subset V$

$$\widetilde{H}_\Lambda^{L,\beta} := -\frac{1}{\beta}\log \mathbb{E}_L[e^{-\beta H_\Lambda}] = \sum_{X \subset \Lambda} \widetilde{\Phi}_X^{L,\beta}\,.$$

We will say that $\widetilde{\Phi}^{L,\beta}$ is the local effective interaction of the marginals $(\rho_\beta^\Lambda)_L$ on L.

Effective Hamiltonian - weak form

Definition

Let us say that the above quantum spin system has (weak) local effective Hamiltonians at (inverse) temperature $\beta > 0$ if it satisfies the following property: for every subset $L \subset V$ there exists a local interaction $\widehat{\Phi}^{L,\beta}$ on V such that

(i) $\widehat{\Phi}^{L,\beta}_X$ is supported in $X \cap L$ for every finite $X \subset V$.

(ii) $\widehat{\Phi}^{L,\beta}_X = \widehat{\Phi}^{L',\beta}_X$ for all finite subset $X \subset V$ and $L' \subset V$ satisfying $X \cap L' = X \cap L$.

(iii) For every finite subset $\Lambda \subset V$

$$\widehat{H}^{L,\beta}_\Lambda := -\frac{1}{\beta} \log\left(\mathrm{tr}_{\Lambda \setminus L}[e^{-\beta H_\Lambda}]\right) + \frac{1}{\beta} \log[Z_{\Lambda\setminus L}]\mathbb{1} = \sum_{X \subset \Lambda, X \cap L \neq \emptyset} \widehat{\Phi}^{L,\beta}_X .$$

For general Hamiltonians, Kuwahara et al. 2019 claimed the existence of suitable weak effective Hamiltonians at high enough temperature using a non-commutative cluster expansion, but the results are faulty (present status unclear)

Why two notions of effective Hamiltonian?

- It can be checked that (ii) implies $\widetilde{\Phi}_X^{V,\beta} = \Phi_X$ for every finite subset $X \subset V$
- We can write

$$-\frac{1}{\beta}\log\left(\mathrm{tr}_{\Lambda\setminus L}[e^{-\beta H_\Lambda}]\right) + \frac{1}{\beta}\log[Z_{\Lambda\setminus L}]\mathbb{1} = -\frac{1}{\beta}\log\left(\mathbb{E}_L[e^{-\beta H_\Lambda}]\right) + \frac{1}{\beta}\log\mathbb{E}_L[e^{-\beta H_{\Lambda\setminus L}}]\mathbb{1}$$

- Can be used to check that the existence of a strong effective Hamiltonian implies that of a weak effective Hamiltonian
- We have neither proof nor counterexample for the reverse implication
- However, we can prove the existence of strong effective Hamiltonians with short range interactions under a Commuting Hypothesis:

Definition

Let us say that a local interaction Φ on V satisfies the Commuting Hypothesis if there is a commuting algebra $\mathcal{A} \subset \mathfrak{A}_V$ such that $\Phi_X \in \mathcal{A}$ for every finite $X \subset V$, and moreover, for every $L \subset V$ the conditional expectation $\mathbb{E}_L[\cdot]$ satisfies $\mathbb{E}_L[\mathcal{A}] \subset \mathcal{A}$.

Existence of strong effective Hamiltonians under the commuting hypothesis

Theorem

Let us consider a quantum spin system with local interaction Φ on V satisfying the Commuting Hypothesis and such that for some $\varepsilon > 0$

$$\|\Phi\|_{\varepsilon,\mathbf{b}} = \sup_{x\in V} \sum_{X\ni x} \|\Phi_X\| e^{\varepsilon|X|+\mathbf{b}(X)} < \infty\,.$$

Then, for every $\beta \in \mathbb{C}$ with $|\beta| \leq \varepsilon/(2\|\Phi\|_{\varepsilon,\mathbf{b}})$ there are (strong) local effective Hamiltonians, namely for every $L \subset V$ there exists a local interaction $\widetilde{\Phi}^{L,\beta}$ on V satisfying (i)-(iii), and moreover

$$\|\widetilde{\Phi}^{L,\beta}\|_{\mathbf{b}} = \sup_{x\in V} \sum_{X\ni x} \|\widetilde{\Phi}^{L,\beta}_X\| e^{\mathbf{b}(X)} < \frac{\varepsilon}{2}\,.$$

For example, $\mathbf{b}(X) = \lambda|X| + \mu\,\mathrm{diam}(X)$.

Proof idea: cluster expansion (works as in the scalar case since everything commutes)

Main result

Theorem

Let Φ *be a local interaction on* $V = \mathbb{Z}^{g}$ *satisfying for some* $\lambda, \mu, \Delta > 0$

$$\|\Phi\|_{\lambda,\mu} = \sup_{x \in V} \sum_{X \ni x} \|\Phi_X\| e^{\lambda |X| + \mu \operatorname{diam}(X)} \leq \Delta\,.$$

Moreover, let $0 < \beta < \lambda/(2\Delta)$ *be an inverse temperature such that:*

- *There is a weak local effective Hamiltonian at temperature* $\beta > 0$*, and for every* $L \subset V$*, the local interaction* $\widehat{\Phi}^{L,\beta}$ *satisfies* $\|\widehat{\Phi}^{L,\beta}\|_{\lambda,\mu} \leq \Delta$.
- Φ *satisfies* $\epsilon(\ell)$*-clustering property.*

Then, there exists constants $\widehat{K}', c' > 0$ *such that for every finite* $\Lambda \subset V$ *and every pair of disjoint subsets* $A, C \subset \Lambda$*, the local Gibbs state* $\rho = \rho_\beta^\Lambda$ *satisfies*

$$\left\| \rho_{AC}\, \rho_A^{-1} \otimes \rho_C^{-1} - \mathbb{1} \right\| \leq \widehat{K}' e^{-c' \operatorname{dist}(A,C)}\,.$$

Moreover, $\widehat{K}' = \mathcal{O}(\min\{e^{|\partial A|}(|\partial A| + |C| g(A)), e^{|\partial C|}(|\partial C| + |A| g(C))\})$.

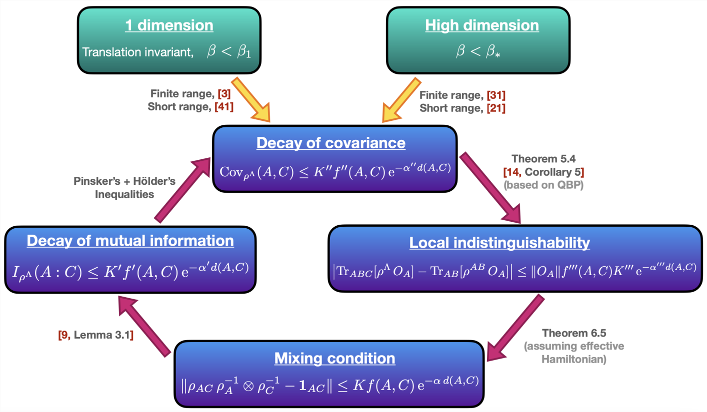
Proof sketch

1. Use effective Hamiltonian to bound

$$\left\|\rho_{AC}\rho_A^{-1}\otimes\rho_C^{-1}-\mathbb{1}_{AC}\right\|_\infty$$
$$\leq\left\|\mathrm{e}^{-\beta\widehat{H}_\Lambda^{AC,\beta}}\,\mathrm{e}^{\beta(\widehat{H}_\Lambda^{A,\beta}+\widehat{H}_\Lambda^{C,\beta})}\right\|_\infty|\kappa_{ABC}-1|+\left\|\mathrm{e}^{-\beta\widehat{H}_\Lambda^{AC,\beta}}\,\mathrm{e}^{\beta(\widehat{H}_\Lambda^{A,\beta}+\widehat{H}_\Lambda^{C,\beta})}-\mathbb{1}_{AC}\right\|_\infty,$$

   where $\kappa_{ABC}=Z_{ABC}Z_BZ_{AB}^{-1}Z_{BC}^{-1}$.
2. Use Araki's expansionals to bound the operator norm terms.
3. Local indistinguishability can be proved from quantum belief propagation. It can in turn be used to estimate $|\kappa_{ABC}-1|$, combined with Araki's expansionals for the original interaction.

Summary



Do we need to assume the existence of an effective Hamiltonian in higher dimensions?

References

Based on:

- AB, Á. Capel, and A. Pérez-Hernández. "Exponential decay of mutual information for Gibbs states of local Hamiltonians". In: *Quantum* 6 (2022), p. 650.
- AB, Á. Capel, and A. Pérez-Hernández. "Strong decay of correlations for Gibbs states in any dimension". In: *arXiv preprint* arXiv:2401.10147 (2024).

Related work:

- H. Araki. "Gibbs states of the one-dimensional quantum spin chain". In: *Commun. Math. Phys.* 14 (1969), pp. 120–157.
- M. Kliesch, C. Gogolin, M. J. Kastoryano, A. Riera, and J. Eisert. "Locality of temperature". In: *Phys. Rev. X* 4 (2014), p. 031019.
- K. Kato and F. G. S. L. Brandão. "Quantum approximate Markov chains are thermal". In: *Commun. Math. Phys.* 370 (2019), pp. 117–149.
- T. Kuwahara, K. Kato, and F. G. S. L. Brandão. "Clustering of conditional mutual information for quantum Gibbs states above a threshold temperature". In: *Phys. Rev. Lett.* 124 (2020), p. 220601.
- A. Capel, M. Moscolari, S. Teufel, and T. Wessel. "From decay of correlations to locality and stability of the Gibbs state". In: *arXiv preprint* arXiv:2310.09182 (2023).