Computing Noise Robustness of Incompatible Quantum Measurements

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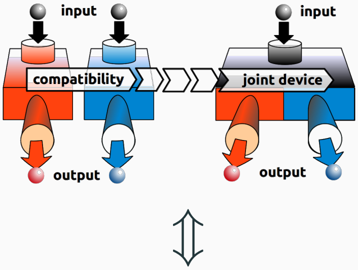
Talk outline

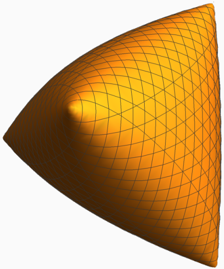
Incompatibility in QM

Free spectrahedra

Connecting the two

Computing the noise robustness of incompatibility





Incompatibility in QM

Quantum states and measurements

- Motivation: Classical state $\rightsquigarrow$ probability distributions: $p \in \mathbb{R}^d$, $p \geq 0$, $\sum_i p_i = 1$
- Quantum states $\rightsquigarrow$ density matrices: $\rho \in \mathcal{M}_d(\mathbb{C})$, $\rho \geq 0$, $\operatorname{Tr} \rho = 1$
- Measurement outcomes are labeled $\{1, \ldots, k\}$, need to be assigned probabilities
- Measurements: Tuples of matrices $(E_1, \ldots, E_k)$ such that $(\operatorname{Tr}[E_1\rho], \ldots, \operatorname{Tr}[E_k\rho])$ is a probability distribution for all states ρ
  - $\operatorname{Tr}[E_i\rho] \in \mathbb{R} \rightsquigarrow E_i = E_i^*$
  - $\operatorname{Tr}[E_i\rho] \geq 0 \rightsquigarrow E_i \geq 0$
  - $\sum_i \operatorname{Tr}[E_i\rho] = 1 \rightsquigarrow \sum_i E_i = I_d$
- Tuples of PSD matrices summing to identity are called positive operator-valued measures (POVMs)
- We call $0 \leq E \leq I$ quantum effects

Quantum measurements: Compatibility

- Quantum measurements $\rightsquigarrow$ give the probabilities of the classical outcomes when a quantum state enters a measurement apparatus. Mathematically, measurements are modeled by POVMs

Definition

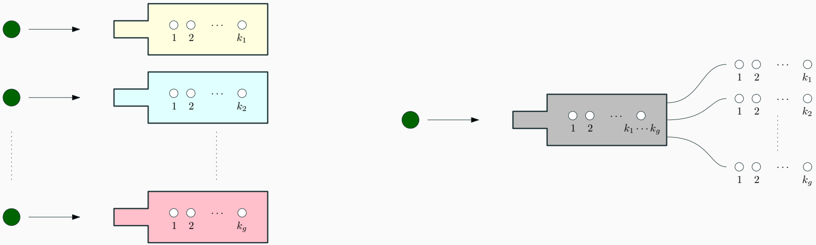
Two POVMs, $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_l)$, are called compatible if there exists a third POVM $C = (C_{ij})_{i \in [k], j \in [l]}$ such that

$$\forall i \in [k], \quad A_i = \sum_{j=1}^{l} C_{ij} \qquad \text{and} \qquad \forall j \in [l], \quad B_j = \sum_{i=1}^{k} C_{ij}$$

The definition generalizes to g-tuples of POVMs $A^{(1)}, \ldots, A^{(g)}$, having respectively $k_1, \ldots k_g$ outcomes, where the joint POVM C has outcome set $[k_1] \times \cdots \times [k_g]$.

- Other way to say that: jointly measurable

What does it mean?



- Compatible measurements can be simulated by a single joint measurement, by classically post-processing its outputs
- Examples:
  1. Trivial POVMs $A = (p_i I_d)$ and $B = (q_j I_d)$ are compatible
  2. Commuting POVMs $[A_i, B_j] = 0$ are compatible
  3. If the POVM A is projective, then A and B are compatible iff they commute

Noisy POVMs

- POVMs can be made compatible by adding noise, i.e. mixing in trivial POVMs
- Example: dichotomic POVMs and white noise, $s \in [0, 1]$

$$(E, I - E) \mapsto s(E, I - E) + (1 - s)(\frac{I}{2}, \frac{I}{2}) \qquad \text{or} \qquad E \mapsto sE + (1 - s)\frac{I}{2}$$

- Taking $s = 1/2$ suffices to render any pair of dichotomic POVMs compatible $\rightsquigarrow$ define $C_{ij} := (E_i + F_j)/4$
- For most of the talk, we focus on dichotomic (YES/NO) POVMs

Definition

The incompatibility degree for g measurements on $\mathbb{C}^d$ is the number

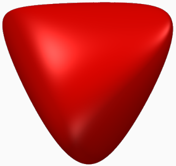
$$\gamma(g, d) := \max\{s \in [0, 1] \,:\, \text{for all quantum effects } E_1, \ldots, E_g \in \mathcal{M}_d(\mathbb{C}), \text{ the noisy versions } sE_i + (1 - s)I_d/2 \text{ are compatible}\}$$

Free spectrahedra

Free spectrahedra

- A spectrahedron is given by PSD constraints: for $A = (A_1, \ldots, A_g) \in (\mathcal{M}_d(\mathbb{C})^{\mathrm{sa}})^g$

$$\mathcal{D}_A(1) := \left\{ x \in \mathbb{R}^g \ : \ \sum_{i=1}^{g} x_i A_i \le I_d \right\}$$



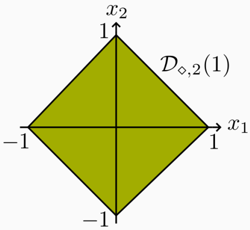
- $\mathcal{D}_{(\sigma_X, \sigma_Y, \sigma_Z)}(1) = \{(x, y, z) \in \mathbb{R}^3 \ : \ x\sigma_X + y\sigma_Y + z\sigma_Z \le I_2\} = \text{Bloch ball}$
- A free spectrahedron is the matricization of a spectrahedron

$$\mathcal{D}_A := \bigsqcup_{n=1}^{\infty} \mathcal{D}_A(n) \quad \text{with} \quad \mathcal{D}_A(n) := \left\{ X \in (\mathcal{M}_n(\mathbb{C})^{\mathrm{sa}})^g \ : \ \sum_{i=1}^{g} X_i \otimes A_i \le I_{nd} \right\}$$

Example: the matrix diamond

The matrix diamond is the free spectrahedron defined by

$$\mathcal{D}_{\diamondsuit,g} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n(\mathbb{C})^{\mathrm{sa}})^g \; : \; \sum_{i=1}^{g} \epsilon_i X_i \leq I_n, \quad \forall \epsilon \in \{\pm 1\}^g\}$$



- At level one, $\mathcal{D}_{\diamondsuit,g}(1)$ is the unit ball of the ℓ^1 norm on $\mathbb{R}^g$
- As a free spectrahedron, it is defined by $2^g \times 2^g$ diagonal matrices $\mathcal{D}_{\diamondsuit,g} = \mathcal{D}_{L_1,\ldots,L_g}$, with $L_i = I_2 \otimes \cdots \otimes I_2 \otimes \mathrm{diag}(1,-1) \otimes I_2 \otimes \cdots \otimes I_2$

Spectrahedral inclusion

- Consider two free spectrahedra defined by $(A_1, \dots, A_g)$ and $(B_1, \dots, B_g)$
- We write $\mathcal{D}_A \subseteq \mathcal{D}_B$ if, for all $n \geq 1$, $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$
- Clearly, $\mathcal{D}_A \subseteq \mathcal{D}_B \implies \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$. For the converse implication to hold, one may need to shrink $\mathcal{D}_A$...

Definition

For a free spectrahedron $\mathcal{D}_A$, we define its inclusion constant as

$$\delta_A(g, d) := \max\{s \in [0,1] : \text{for all } g\text{-tuples } B_1, \dots, B_g \in \mathcal{M}_d(\mathbb{C})^{\mathrm{sa}}, \\ \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1) \implies s.\mathcal{D}_A \subseteq \mathcal{D}_B\}$$

- We shall be concerned with the inclusion constant for the matrix diamond, which we denote by $\delta(g, d)$

Connecting the two

Compatibility in QM $\iff$ matrix diamond inclusion

To a g-tuple $E \in (\mathcal{M}_d(\mathbb{C})^{\text{sa}})^g$, we associate:

$$\mathcal{D}_{2E-I} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n(\mathbb{C})^{\text{sa}})^g \ : \ \sum_{i=1}^{g} X_i \otimes (2E_i - I_d) \leq I_{nd}\}$$

Theorem

Let $E \in (\mathcal{M}_d(\mathbb{C})^{sa})^g$ be g-tuple of selfadjoint matrices. Then:

- *The matrices E are quantum effects $\iff \mathcal{D}_{\diamondsuit,g}(1) \subseteq \mathcal{D}_{2E-I}(1)$*
- *The matrices E are compatible quantum effects $\iff \mathcal{D}_{\diamondsuit,g} \subseteq \mathcal{D}_{2E-I}$*

At the intermediate levels $1 \leq n \leq d$, $\mathcal{D}_{\diamondsuit,g}(n) \subseteq \mathcal{D}_{2E-I}(n)$ iff for all isometries $V : \mathbb{C}^n \to \mathbb{C}^d$, the compressed effects $V^* E_i V$ are compatible.

Moreover, the incompatibility degree is equal to the inclusion constant of the matrix diamond: $\forall g, d$, $\gamma(g,d) = \delta(g,d)$.

Consequences

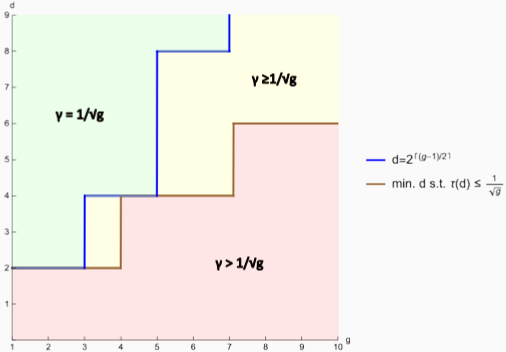
Many things are known about the matrix diamond:

- For all g, d, $\tau(d) \leq \delta(g, d)$, $\tau(d) \approx \sqrt{\frac{2}{\pi d}}$ asymptotically (AB and Nechita, 2022)
- For all g, d, $\frac{1}{\sqrt{g}} \leq \delta(g, d)$ (Passer *et al.*, 2018)

Theorem (Passer et al., 2018)

For all g and $d \geq 2^{\lceil (g-1)/2 \rceil}$, $\gamma(g, d) = \delta(g, d) = \frac{1}{\sqrt{g}}$

Phase diagram



- Connection to free spectrahedra also holds for arbitrary outcomes
- Instead of matrix diamond, consider its generalization, the matrix jewel
- The smallest unknown case is $d = 2$, $g = 4$ (4 qubit measurements)

Computing the noise robustness of incompatibility

4 qubit measurements: Reminder

- Connection between compatibility and free spectrahedra: The matrices E are compatible quantum effects $\iff \mathcal{D}_{\diamondsuit,g} \subseteq \mathcal{D}_{2E-I}$
- $B_i = 2E_i - I$ for some $I \geq E_i \geq 0 \iff -I \leq B_i \leq I$
- $X \in \mathcal{D}_{\diamondsuit,4}(2) \iff \sum_{i=1}^{4} \epsilon_i X_i \leq I_2 \quad \forall \epsilon \in \{\pm 1\}^4$
- $X \in \mathcal{D}_{2E-I} \iff \sum_{i=1}^{4} B_i \otimes X_i \leq I$

Goal: Maximize

$$\lambda_{\max}\left(\sum_{i=1}^{4} B_i \otimes X_i\right)$$

over all allowed X, B as above.

Value: $1/\delta(4,2) = 1/\gamma(4,2)$

4 qubit measurements: Optimization problem

$$
\begin{aligned}
\text{maximize} \quad & \lambda_{\max}\left(\sum_{i=1}^{4} B_i \otimes X_i\right) \\
\text{subject to} \quad & B_1 \leq I_2, \quad -B_1 \leq I_2, \\
& B_2 \leq I_2, \quad -B_2 \leq I_2, \\
& B_3 \leq I_2, \quad -B_3 \leq I_2, \\
& B_4 \leq I_2, \quad -B_4 \leq I_2, \\
& \sum_{i=1}^{4} \epsilon_i X_i \leq I_2, \qquad \forall \epsilon \in \{\pm 1\}^4 \\
& B_i,\ X_i \in \mathcal{M}_2(\mathbb{C})^{\mathrm{sa}} \qquad \forall i \in \{1,2,3,4\}
\end{aligned}
$$

Problem: We need to keep the dimension fixed, otherwise the result is $1/2$

Bloch sphere expansion

- Use Pauli matrices to write

$$B_i = b_0^{(i)} I_2 + b_1^{(i)} \sigma_X + b_2^{(i)} \sigma_Y + b_3^{(i)} \sigma_Z$$
$$X_i = x_0^{(i)} I_2 + x_1^{(i)} \sigma_X + x_2^{(i)} \sigma_Y + x_3^{(i)} \sigma_Z.$$

- We obtain a polynomial optimization problem in commuting variables
- Solve it with Lasserre-Parillo hierarchy

Commutative polynomial optimization problem

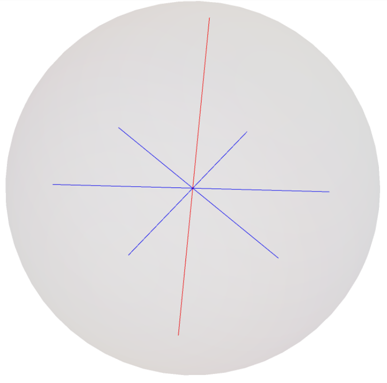
$$
\begin{aligned}
\text{maximize} \quad & \sum_i \sum_j b_j^{(i)} x_j^{(i)} \\
\text{subject to} \quad & 1 - (b_1^{(0)})^2 - (b_2^{(0)})^2 - (b_3^{(0)})^2 \geq 0, \\
& (1 - b_0^{(i)})^2 - (b_1^{(0)} - b_1^{(i)})^2 - (b_2^{(0)} - b_2^{(i)})^2 - (b_3^{(0)} - b_3^{(i)})^2 \geq 0, \quad \forall i \in [4] \\
& (1 + b_0^{(i)})^2 - (b_1^{(0)} + b_1^{(i)})^2 - (b_2^{(0)} + b_2^{(i)})^2 - (b_3^{(0)} + b_3^{(i)})^2 \geq 0, \quad \forall i \in [4] \\
& 1 \geq b_0^{(i)} \geq -1, \qquad \forall i \in \{1, 2, 3, 4\} \\
& \Big(1 - \sum_i \epsilon_i x_0^{(i)}\Big)^2 \geq \sum_j \Big(\sum_i \epsilon_i x_j^{(i)}\Big)^2, \qquad \forall \epsilon \in \{\pm 1\}^4 \\
& 1 \geq \sum_i \epsilon_i x_0^{(i)}, \qquad \forall \epsilon \in \{\pm 1\}^4 \\
& b_j^{(i)},\ x_j^{(i)} \in \mathbb{R} \qquad \forall i, j
\end{aligned}
$$

Conjectured noise robustness of 4 qubit measurements

Known constraints:

- $\gamma(g, d) \geq 1/\sqrt{g} \implies \gamma(4, 2) \geq 1/2$
- $\gamma(g, d) \geq \gamma(g+1, d)$ and $\gamma(3, 2) = 1/\sqrt{3}$
  $\implies \gamma(4, 2) \leq 1/\sqrt{3} \approx 0.58$

Conjecture: $\gamma(4, 2) = 2/\sqrt{13} \approx 0.55$



- Theory of extreme points for free spectrahedra: $\gamma(4, 2) \leq 2/\sqrt{13}$
- Polynomial optimization: Almost matching lower bounds

More outcomes

- The connection between compatibility and free spectrahedra also holds for more outcomes
- Need to substitute the matrix diamond by the matrix jewel
- Can for example consider one measurement with 2 and one measurement with k outcomes
- Not much is known for this problem
- Can write down a polynomial optimization problem again to compute the noise robustness

Polynomial optimization problem for $2+3$ outcomes

$$
\begin{aligned}
\text{maximize} \quad & \lambda_{\max}\left(\sum_{i=1}^{3} B_i \otimes X_i\right) \\
\text{subject to} \quad & B_1 \leq I_m, \quad -B_1 \leq I_m, \\
& -\frac{3}{2}B_2 \leq I_m, \quad -\frac{3}{2}B_3 \leq I_m, \\
& \frac{3}{2}B_2 + \frac{3}{2}B_3 \leq I_m, \\
& \pm X_1 + \frac{4}{3}X_2 - \frac{2}{3}X_3 \leq I_m, \\
& \pm X_1 - \frac{2}{3}X_2 + \frac{4}{3}X_3 \leq I_m, \\
& \pm X_1 - \frac{2}{3}X_2 - \frac{2}{3}X_3 \leq I_m, \\
& B_i,\ X_i \in \mathcal{M}_m(\mathbb{C})^{\mathrm{sa}} \qquad \forall i \in \{1,2,3\}
\end{aligned}
$$

If we don't fix the dimension m, we can solve this using the NPA hierarchy or the hierarchy for solving trace polynomials by Klep *et al.*, 2022

Conjectured noise robustness of $2 + k$ outcomes measurements

- We know that $1/2 \leq \gamma((2,3), m) \leq 1/\sqrt{2}$

Conjecture: $\gamma((2,k), m) = \frac{1}{2}\left(1 + \frac{1}{1+\sqrt{k}}\right)$

For $k \in \{2, 3, 4, 5\}$:

- Theory of extreme points for free spectrahedra: $\gamma((2,k), m) \leq \frac{1}{2}\left(1 + \frac{1}{1+\sqrt{k}}\right)$
- Corresponding measurements: $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ and $\{|+\rangle\langle +|, |-\rangle\langle -|, 0, \dots, 0\}$
- NPA hierarchy almost matching lower bounds up to high precision

Open questions

- Can we simplify the optimization problems further using knowledge about extreme points of free spectrahedra?
- Can we extract exact certificates from the polynomial optimization?
- Can we solve the polynomial optimization problem more efficiently (allowing to tackle larger problems)?

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