# Unified framework for continuity of sandwiched Rényi divergences

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# Motivation

- Well-known continuity bound: Alicki-Fannes-Winter

$$|H_\rho(A|B) - H_\sigma(A|B)| \leq 2\varepsilon \log d_A + (1+\varepsilon) h\Big(\frac{\varepsilon}{1+\varepsilon}\Big)\,.$$

  with $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon \leq 1$ and $h$ the binary entropy
- More generally: Give bounds on

$$\sup\{|f(\rho) - f(\sigma)| : \rho, \sigma \in \mathcal{S}_0, d(\rho, \sigma) \leq \varepsilon\}$$

  for some entropic quantities $f$
- Especially useful if you know the value of the entropic quantity for some states, but not for others.

Aim: Find methods that give good continuity bounds for as many entropic quantities as possible

# Sandwiched Rényi entropies

- This talk: entropic quantities derived from sandwiched Rényi entropies
$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log\Big(\widetilde{Q}_\alpha(\rho\|\sigma)\Big) = \frac{1}{\alpha-1}\log\operatorname{tr}\Big[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha\Big],$$
where $\alpha \in [1/2,1)\cup(1,\infty)$. Limit $\alpha \to 1$ yields relative entropy.
- Examples of such quantities:
$$\widetilde{H}^\uparrow_\alpha(A|B)_\rho := \sup_{\tau_B\in\mathcal{S}(\mathcal{H}_B)} -\widetilde{D}_\alpha(\rho_{AB}\|\mathbb{1}_A\otimes\tau_B)$$
and
$$\widetilde{I}^\uparrow_\alpha(A:B)_\rho = \inf_{\substack{\tau_A\in\mathcal{S}(\mathcal{H}_A),\\ \tau_B\in\mathcal{S}(\mathcal{H}_B)}} \widetilde{D}_\alpha(\rho_{AB}\|\tau_A\otimes\tau_B)$$

# Previous results

- We were inspired by two previous works for $\widetilde{H}_\alpha^\uparrow(A|B)_\rho$:
- Marwah and Dupuis [MD22]:

$$\left|\widetilde{H}_\alpha^\uparrow(A|B)_\rho - \widetilde{H}_\alpha^\uparrow(A|B)_\sigma\right| \leq \log(1+\varepsilon) + \frac{1}{1-\alpha}\log\left(1+\varepsilon^\alpha d_A^{2(1-\alpha)} - \frac{\varepsilon}{(1+\varepsilon)^{1-\alpha}}\right) \qquad \alpha \in [1/2, 1)$$

$$\left|\widetilde{H}_\alpha^\uparrow(A|B)_\rho - \widetilde{H}_\alpha^\uparrow(A|B)_\sigma\right| \leq \log\left(1+\sqrt{2\varepsilon}\right) + \frac{1}{1-\beta}\log\left(1+\sqrt{2\varepsilon}^\beta d_A^{2(1-\beta)} - \frac{\sqrt{2\varepsilon}}{(1+\sqrt{2\varepsilon})^{1-\beta}}\right) \qquad \alpha \in (1, \infty)$$

  where $\beta$ is such that $\alpha^{-1} + \beta^{-1} = 2$
- Beigi and Goodarzi [BG23]:

$$\left|\widetilde{H}_\alpha^\uparrow(A|B)_\rho - \widetilde{H}_\alpha^\uparrow(A|B)_\sigma\right| \leq \alpha' \log\left(1 + 2\varepsilon d_A^{2/\alpha'}\right)$$

  where $\alpha' = \alpha/(\alpha-1)$
- From these, we build three approaches: almost additive, operator space, and mixed approach

# Almost additive approach: Ingredients

- This approach builds on the ideas in [MD22]
- Uses properties of $\widetilde{Q}_\alpha(\rho\|\sigma) = \mathrm{tr}\left[(\sigma^{\frac{1:-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha\right]$:
  1. $(\rho,\sigma) \mapsto \widetilde{Q}_\alpha(\rho\|\sigma)$ is jointly concave for $\alpha \in [1/2, 1)$ and jointly convex for $\alpha \in (1, \infty)$.
  2. Let $X_i$, $Y$ be positive operators with suitable supports. For $\alpha \in (0, 1)$

     $$\widetilde{Q}_\alpha(X_1 + X_2\|Y) \leq \widetilde{Q}_\alpha(X_1\|Y) + \widetilde{Q}_\alpha(X_2\|Y)$$

     and for $\alpha \in (1, \infty)$

     $$\widetilde{Q}_\alpha(X_1\|Y) + \widetilde{Q}_\alpha(X_2\|Y) \leq \widetilde{Q}_\alpha(X_1 + X_2\|Y)\,.$$

- We can convert these properties into continuity bounds for

  $$\widetilde{D}_{\alpha,\mathcal{C}} : \mathcal{S}(\mathcal{H}) \to \mathbb{R}, \qquad \rho \mapsto \widetilde{D}_{\alpha,\mathcal{C}}(\rho) := \inf_{\tau\in\mathcal{C}} \widetilde{D}_\alpha(\rho\|\tau)$$

  when $\mathcal{C} \subseteq \mathcal{S}(\mathcal{H})$ is compact, convex set containing at least one positive definite state

# Almost additive approach: Result

## Theorem (Distance to a compact, convex set)

*Let* $\mathcal{C} \subseteq \mathcal{S}(\mathcal{H})$ *be a compact, convex set that contains at least one positive definite state. For* $\rho$, $\sigma \in \mathcal{S}(\mathcal{H})$ *with* $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$, $\alpha \in [1/2, 1)$ *and* $\kappa$ *such that* $\sup_{\rho \in \mathcal{S}(\mathcal{H})} \widetilde{D}_{\alpha,\mathcal{C}}(\rho) \leq \log(\kappa) < \infty$ *we get*

$$|\widetilde{D}_{\alpha,\mathcal{C}}(\rho) - \widetilde{D}_{\alpha,\mathcal{C}}(\sigma)| \leq \log(1+\varepsilon) + \frac{1}{1-\alpha} \log\left(1 + \varepsilon^{\alpha}\kappa^{1-\alpha} - \frac{\varepsilon}{(1+\varepsilon)^{1-\alpha}}\right)$$

*Further for* $\alpha \in (1, \infty)$ *and* $\kappa$ *such that* $\sup_{\rho \in \mathcal{S}(\mathcal{H})} \widetilde{D}_{\alpha,\mathcal{C}}(\rho) \leq \log(\kappa) < \infty$

$$|\widetilde{D}_{\alpha,\mathcal{C}}(\rho) - \widetilde{D}_{\alpha,\mathcal{C}}(\sigma)| \leq \log(1+\varepsilon) + \frac{1}{\alpha-1} \log\left(1 + \varepsilon\kappa^{\alpha-1} - \frac{\varepsilon^{\alpha}}{(1+\varepsilon)^{\alpha-1}}\right).$$

## Operator space approach: Ingredients (1/2)

- This approach builds on the ideas in [BG23]

**Definition (The $\mathcal{C}, p, q$ norm)**

Let $\mathcal{C} \subset \mathcal{B}_{\geq 0}(\mathcal{H})$ be a compact, convex set containing at least one positive definite state. Then for $1 \leq p \leq q \leq \infty$, $\frac{1}{r} := \frac{1}{p} - \frac{1}{q}$, we define

$$\|\cdot\|_{\mathcal{C},p,q} : \mathcal{B}(\mathcal{H}) \to [0,\infty), \quad X \mapsto \|X\|_{\mathcal{C},p,q} := \sup_{c \in \mathcal{C}} \left\| c^{\frac{1}{2r}} X c^{\frac{1}{2r}} \right\|_p .$$

- Usually, $\mathcal{C}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, but we consider compact convex subsets of $\mathcal{B}(\mathcal{H})$ consisting of positive semidefinite operators containing at least one full-rank state
- For $1 \leq q' \leq p' \leq \infty$ such that $\frac{1}{r} = \frac{1}{q'} - \frac{1}{p'}$, define

$$\|\cdot\|^*_{\mathcal{C},p',q'} : \mathcal{B}(\mathcal{H}) \to [0,\infty), \quad X \mapsto \|X\|^*_{\mathcal{C},p',q'} := \inf_{c \in \mathcal{C}, c > 0} \left\| c^{-\frac{1}{2r}} X c^{-\frac{1}{2r}} \right\|_{p'} .$$

- Not clear that this is a norm

# Operator space approach: Ingredients (2/2)

- We prove (without interpolation theory) that the dual quantity is subadditive on positive semi-definite elements, i.e.,

$$\|X+Y\|^*_{\mathcal{C},p',q'} \leq \|X\|^*_{\mathcal{C},p',q'} + \|Y\|^*_{\mathcal{C},p',q'}$$

for $X, Y \in \mathcal{B}_{\geq 0}(\mathcal{H})$.
- Intermediate steps:
  1. Hölder inequality: $|\operatorname{tr}[XY]| \leq \|X\|_{\mathcal{C},p,q}\|Y\|^*_{\mathcal{C},p',q'}$
  2. For $X \in \mathcal{B}(\mathcal{H})$, we find

$$\sup_{Y\in\mathcal{B}(\mathcal{H}),\, \|Y\|^*_{\mathcal{C},p',q'}\leq 1} |\operatorname{tr}[XY]| = \|X\|_{\mathcal{C},p,q}$$

  3. For $X \geq 0$, we find

$$\sup_{Y\in\mathcal{B}_{\geq 0}(\mathcal{H}),\|Y\|_{\mathcal{C},p,q}\leq 1} |\operatorname{tr}[XY]| = \|X\|^*_{\mathcal{C},p',q'}$$

# Operator space approach: Result

**Theorem (Distance to convex, compact set)**

*Let $\mathcal{C} \subseteq \mathcal{S}(\mathcal{H})$ be a compact, convex set that contains at least one positive definite state. For $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ with $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$, $\alpha \in [1/2, 1)$ and $\kappa$ such that* $\sup_{\rho \in \mathcal{S}(\mathcal{H})} \widetilde{D}_{\alpha,\mathcal{C}}(\rho) \leq \log(\kappa) < \infty$,

$$|\widetilde{D}_{\alpha,\mathcal{C}}(\rho) - \widetilde{D}_{\alpha,\mathcal{C}}(\sigma)| \leq \frac{1}{1-\alpha} \log\left(1 + \varepsilon^{\alpha}\kappa^{1-\alpha}\right).$$

*Further for $\alpha \in (1, \infty)$ and $\kappa$ such that* $\sup_{\rho \in \mathcal{S}(\mathcal{H})} \widetilde{D}_{\alpha,\mathcal{C}}(\rho) \leq \log(\kappa) < \infty$ *we have*

$$|\widetilde{D}_{\alpha,\mathcal{C}}(\rho) - \widetilde{D}_{\alpha,\mathcal{C}}(\sigma)| \leq \frac{\alpha}{\alpha - 1} \log\left(1 + \varepsilon\kappa^{\frac{\alpha-1}{\alpha}}\right).$$

# Mixed approach: Result

- The mixed approach combines ideas from both previous ones

**Theorem (Distance to convex, compact set)**

*Let $\mathcal{C} \subseteq \mathcal{S}(\mathcal{H})$ be a compact, convex set that contains at least one positive definite state. For $\rho$, $\sigma \in \mathcal{S}(\mathcal{H})$ satisfying $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$ and $\kappa$ such that $\sup_{\rho \in \mathcal{S}(\mathcal{H})} \widetilde{D}_{\alpha,\mathcal{C}}(\rho) \leq \log(\kappa) < \infty$ we find*

$$|\widetilde{D}_{\alpha,\mathcal{C}}(\rho) - \widetilde{D}_{\alpha,\mathcal{C}}(\sigma)| \leq \log(1+\varepsilon) + \frac{\alpha}{\alpha - 1} \log\left(1 + \varepsilon\kappa^{\frac{\alpha-1}{\alpha}} - \frac{\varepsilon^{\frac{2\alpha-1}{\alpha}}}{(1+\varepsilon)^{\frac{\alpha-1}{\alpha}}}\right).$$

# Example: Sandwiched Rényi conditional entropy

**Corollary**

*Let* $\rho, \sigma \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$*, with* $\frac{1}{2}\|\rho - \sigma\| \leq \varepsilon$*, then for* $\alpha \in [1/2, 1)$

$$|\widetilde{H}^{\uparrow}_{\alpha}(A|B)_{\rho} - \widetilde{H}^{\uparrow}_{\alpha}(A|B)_{\sigma}| \leq \log(1+\varepsilon) + \frac{1}{1-\alpha}\log\left(1 + \varepsilon^{\alpha} d_A^{2(1-\alpha)} - \frac{\varepsilon}{(1+\varepsilon)^{1-\alpha}}\right),$$

*which is the bound from [MD22], and for* $\alpha \in (1, \infty)$

$$|\widetilde{H}^{\uparrow}_{\alpha}(A|B)_{\rho} - \widetilde{H}^{\uparrow}_{\alpha}(A|B)_{\sigma}| \leq \min \begin{cases} \log(1+\varepsilon) + \frac{1}{\alpha-1}\log\left(1 + \varepsilon d_A^{2(\alpha-1)} - \frac{\varepsilon^{\alpha}}{(1+\varepsilon)^{\alpha-1}}\right), \\ \frac{\alpha}{\alpha-1}\log\left(1 + \varepsilon d_A^{2\frac{\alpha-1}{\alpha}}\right), \\ \log(1+\varepsilon) + \frac{\alpha}{\alpha-1}\log\left(1 + \varepsilon d_A^{2\frac{\alpha-1}{\alpha}} - \frac{\varepsilon^{2-\frac{1}{\alpha}}}{(1+\varepsilon)^{\frac{\alpha-1}{\alpha}}}\right). \end{cases}$$

# Proof sketch

- $\mathcal{C} = \{d_A^{-1}\mathbb{1}_A \otimes \sigma_B \ : \ \sigma_B \in \mathcal{S}(\mathcal{H}_B)\}$ is a convex and compact set
- Contains the maximally mixed state as positive definite state
- We can rewrite $\widetilde{H}_\alpha^{\uparrow}(A|B)_\rho = -\widetilde{D}_{\alpha,\mathcal{C}}(\rho) + \log d_A$
- Verify that $\sup_{\rho\in\mathcal{S}(\mathcal{H})} \widetilde{D}_{\alpha,\mathcal{C}}(\rho) \leq 2\log d_A$, this gives $\kappa = d_A^2$
- Now we can apply our previous theorems

# Example: First entry of divergence

**Corollary (Continuity bound in the first argument)**

*Let* $\rho, \sigma, \tau \in \mathcal{S}(\mathcal{H})$ *be quantum states, with* $\ker\tau \subseteq \ker\rho \cap \ker\sigma$, $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$ *and* $\alpha \in [1/2, 1)$. *Then*

$$|\widetilde{D}_\alpha(\rho\|\tau) - \widetilde{D}_\alpha(\sigma\|\tau)| \leq \log(1+\varepsilon) + \frac{1}{1-\alpha}\log\left(1 + \varepsilon^\alpha \widetilde{m}_\tau^{\alpha-1} - \frac{\varepsilon}{(1+\varepsilon)^{1-\alpha}}\right),$$

*where* $\widetilde{m}_\tau$ *is the smallest non-zero eigenvalue of* $\tau$. *For* $\alpha \in (1, \infty)$ *we find*

$$|\widetilde{D}_\alpha(\rho\|\tau) - \widetilde{D}_\alpha(\sigma\|\tau)| \leq \min\begin{cases} \log(1+\varepsilon) + \frac{1}{\alpha-1}\log\left(1 + \varepsilon\widetilde{m}_\tau^{1-\alpha} - \frac{\varepsilon^\alpha}{(1+\varepsilon)^{\alpha-1}}\right), \\ \frac{\alpha}{\alpha-1}\log\left(1 + \varepsilon\widetilde{m}_\tau^{\frac{1-\alpha}{\alpha}}\right), \\ \log(1+\varepsilon) + \frac{\alpha}{\alpha-1}\log\left(1 + \varepsilon\widetilde{m}_\tau^{\frac{1-\alpha}{\alpha}} - \frac{\varepsilon^{2-\frac{1}{\alpha}}}{(1+\varepsilon)^{\frac{\alpha-1}{\alpha}}}\right). \end{cases}$$

The proof is a straightforward consequence of our theorems.

# Example: Sandwiched Rényi mutual information

**Corollary**

*Let* $\rho, \sigma \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ *with* $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$. *Then, for* $\alpha \in [1/2, 1)$ *we find*

$$\begin{aligned} &|\widetilde{I}_\alpha(A:B)_\rho - \widetilde{I}_\alpha(A:B)_\sigma| \\ &\quad \leq 2\log\left(1+\varepsilon^{\frac{1}{\alpha}}\right) + \frac{1}{1-\alpha}\log\left(1+\varepsilon^\alpha m^{2(1-\alpha)} - \frac{\varepsilon^{\frac{1}{\alpha}}}{(1+\varepsilon^{\frac{1}{\alpha}})^{2(1-\alpha)}}\right), \end{aligned}$$

*and for* $\alpha \in (1, \infty)$ *we have*

$$\begin{aligned} &|\widetilde{I}_\alpha(A:B)_\rho - \widetilde{I}_\alpha(A:B)_\sigma| \\ &\quad \leq 2\log\left(1+\varepsilon^{\frac{1}{\alpha}}\right) + \frac{1}{\alpha-1}\log\left(1+\varepsilon^{\frac{1}{\alpha}} m^{2(\alpha-1)} - \frac{\varepsilon^{\alpha}}{(1+\varepsilon^{\frac{1}{\alpha}})^{2(\alpha-1)}}\right), \end{aligned}$$

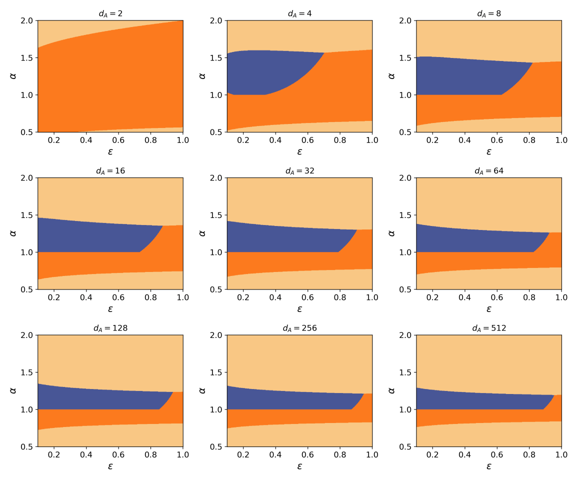
*where in both bounds* $m = \min\{d_A, d_B\}$.

Not straightforward, because we optimize in two arguments. However, the ideas from the almost additive method still work.

# Limits

| | **Approach** | $\alpha \to 1$ | $\alpha \to \infty$ |
|---|---|---|---|
| Conditional Entropy (5.1) | A. additive | $2\varepsilon \log d_A + (1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right)$ | $\log(1+\varepsilon) + 2\log d_A$ |
| | Op. space | $\infty$ | $\log\left(1+\varepsilon d_A^2\right)$ |
| | Mixed | $2\varepsilon \log d_A + (1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right)$ | $\log\left(1+\varepsilon d_A^2 + \varepsilon(1+\varepsilon(d_A^2-1))\right)$ |
| Mutual Info. (5.2) | A. additive | $2\varepsilon \log m + 2(1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right)$ | $\log 4m^2$ |
| 1st Entry of Divergence (5.4) | A. additive | $\varepsilon \log\left(\widetilde{m}_\tau^{-1}\right) + (1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right)$ | $\log(1+\varepsilon) + \log\left(\widetilde{m}_\tau^{-1}\right)$ |
| | Op. space | $\infty$ | $\log\left(1+\varepsilon \widetilde{m}_\tau^{-1}\right)$ |
| | Mixed | $\varepsilon \log\left(\widetilde{m}_\tau^{-1}\right) + (1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right)$ | $\log\left(1+\varepsilon \frac{1}{\widetilde{m}_\tau} + \varepsilon(1+\varepsilon(\frac{1}{\widetilde{m}_\tau}-1))\right)$ |

# Comparison of the approaches




A comparison of the continuity bounds for $\tilde{H}_\alpha(A|B)_\rho$ proven by the almost additive, operator space, and mixed approach.

Every approach performs best in some regime.

# Interlude: Quantum Markov states

- Quantum Markov state: $\rho$ such that $I(A:C|B)_\rho = 0$
- Equivalent to recovery condition

$$\rho_{ABC} = \rho_{BC}^{1/2} \rho_B^{-1/2} \rho_{AB} \rho_B^{-1/2} \rho_{BC}^{1/2} \tag{1}$$

- Can define sandwiched Rényi conditional mutual information as

$$\widetilde{I}_\alpha^{\uparrow}(A:C|B)_\rho := \widetilde{H}_\alpha^{\uparrow}(C|B)_\rho - \widetilde{H}_\alpha^{\uparrow}(C|AB)_\rho$$

- Turns out that $\widetilde{I}_\alpha^{\uparrow}(A:C|B)_\rho$ for $\alpha \in (1/2, 1) \cup (1, \infty)$ if and only if Eq. (1) holds (Jenčová [Jen17], Gao and Wilde [GW21])

> Can we lower bound the distance from being recoverable by the sandwiched Rényi CMI using continuity bounds?

# $\alpha$-approximate quantum Markov chains

**Theorem**

*Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ be positive definite. Given $\alpha \in (1/2, 1) \cup (1, \infty)$, $\rho_{ABC}$ is an $\alpha$-approximate quantum Markov chain if, and only if, it is close to its (rotated, universal) Petz recovery. More specifically, we have for $\alpha \in (1/2, 1)$*

$$\frac{\alpha}{1-\alpha} \log \left( 1 + \left( \mathcal{K} \left\| \rho_{ABC} - \rho_{AB}^{\frac{1}{2}+it} \rho_B^{-\frac{1}{2}-it} \rho_{BC} \rho_B^{-\frac{1}{2}-it} \rho_{AB}^{\frac{1}{2}+it} \right\|_1 \right)^{\frac{1}{1-\frac{1}{2\alpha}-\varepsilon}} \right)$$

$$\leq \widetilde{I}_\alpha(A:C|B)_\rho$$

$$\leq c\left(\alpha, \left\|\rho_{ABC}^{-1}\right\|_\infty^{-1}, d_C, d_{ABC}\right) \left\| \rho_{ABC} - \rho_{BC}^{1/2} \rho_B^{-1/2} \rho_{AB} \rho_B^{-1/2} \rho_{BC}^{1/2} \right\|_1^{1/2},$$

*for any $\varepsilon \in (0, 1 - \frac{1}{2\alpha})$, with $\mathcal{K} = \mathcal{K}(\epsilon, \alpha)$ and similar results for $\alpha \in (1, \infty)$.*

The lower bounds follow from previous work by Gao and Wilde [GW21]

# Proof sketch

- Bound $\widetilde{I}_\alpha^\uparrow(A:C|B)_\rho - \widetilde{I}_\alpha^\uparrow(A:C|B)_\sigma$, where $\sigma$ is a quantum Markov state
- Problem: $\widetilde{I}_\alpha^\uparrow(A:C|B)$ is not of the form $\widetilde{D}_{\alpha,\mathcal{C}}(\rho)$
- Have to resort to a different proof technique, based on the proof of the Alicki-Fannes-Winter bound [Win16], later extended by Shirokov [Shi20] and by AB, Capel, Gondolf, and Pérez-Hernández [BCG+23]
- ALAFF method in [BCG+23] needs almost concavity/almost convexity of

$$(\rho,\sigma) \mapsto \widetilde{Q}_\alpha(\rho\|\sigma)$$

- You could also use the ALAFF method for the quantities we have considered so far, but the bounds from the other three methods are better

# Open questions

- Is $X \mapsto \|X\|^*_{\mathcal{C},p',q'}$ a norm?
- How tight are these bounds?
- Can we find an approach that outperforms all our three approaches?
- Can these continuity bounds be extended to infinite dimensions?
- What about other divergences? (work in progress)

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Based on arXiv:2308.12425


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