Compatibility of quantum measurements and inclusion constants for free spectrahedra

Andreas Bluhm

with Ion Nechita

Technical University of Munich
Department of Mathematics

Madrid, May 15, 2019

Introduction

Compatibility of quantum measurements:

- Measurement = POVM
- Compatible if marginals of common measurement
- Only incompatible measurements can violate Bell inequalities
- Noise robustness quantifies incompatibility

Inclusion of free spectrahedra:

- Convex optimization
- Free spectrahedron = relaxation of linear matrix inequalities (dual SDPs)
- Inclusion constants quantify error

Aim of this talk: Connecting the two problems

Compatibility: Two binary measurements

Example

Consider two binary measurements: $\{E, I - E\}$, $\{F, I - F\}$.
Assume that there is a measurement $\{R_{i,j}\}_{i,j=0}^{1}$ such that

$$
\begin{array}{ccccc}
R_{0,0} & + & R_{0,1} & = & E \\
+ & & + & & \\
R_{1,0} & + & R_{1,1} & = & I - E \\
\| & & \| & & \\
F & & I - F & &
\end{array}
$$

Then the measurements are jointly measurable or compatible.

- For concrete measurements, this can be checked using an SDP.
- There is an equivalent definition via classical post processing.

The compatibility region

- Measurements can be made compatible by adding a sufficient amount of noise
- White noise:

$$E \mapsto sE + \frac{1-s}{2} I_d$$

- Compatibility region:

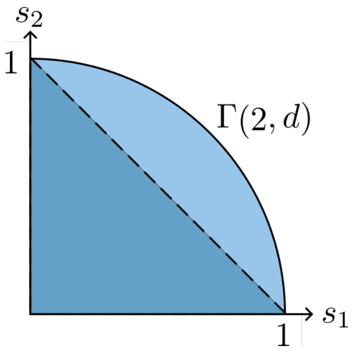
$$\Gamma(g,d) := \Big\{ s \in [0,1]^g : \; s_i E_i + \frac{1-s_i}{2} I_d \text{ are compatible} \\ \forall E_1, \ldots, E_g \in [0, I_d] \Big\}$$

- Incompatibility is a resource for quantum information processing
- Noise robustness can be used to quantify incompatibility
- Lower bounds on $\Gamma(g,d)$ through approximate cloning

An easy example

Example

As $\Gamma(g,d)$ is convex, it holds $\left(\frac{1}{g},\dots,\frac{1}{g}\right) \in \Gamma(g,d)\ \forall d \in \mathbb{N}$



$$\Gamma(g,d) := \Big\{ s \in [0,1]^g : \\ s_i E_i + \frac{1-s_i}{2} I \text{ are comp.} \\ \forall E_1,\dots,E_g \in [0, I_d] \Big\}$$

Free spectrahedra

Let $A \in (M_d^{sa})^g$. The free spectrahedron at level n is defined as

$$\mathcal{D}_A(n) := \left\{ X \in \left(\mathcal{M}_n^{sa}\right)^g : \sum_{i=1}^{g} A_i \otimes X_i \leq I_{nd} \right\}.$$

The free spectrahedron is the union of these levels

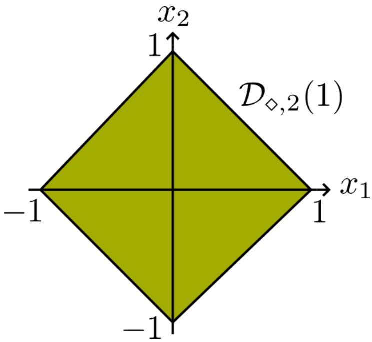
$$\mathcal{D}_A := \bigcup_{n \in \mathbb{N}} \mathcal{D}_A(n).$$

Different free spectrahedra can usually have the same first level $\mathcal{D}_A(1)$.

The matrix diamond

An important example is the matrix diamond:

$$\mathcal{D}_{\diamond,g}(n) = \left\{ X \in \left(\mathcal{M}_n^{sa}\right)^g : \sum_{i=1}^{g} \epsilon_i X_i \leq I_n \ \forall \epsilon \in \{-1,+1\}^g \right\}.$$



Example

For $g = 2$:

$$A_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

Inclusion of free spectrahedra

- $\mathcal{D}_A \subseteq \mathcal{D}_B$ means $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$ for all n

Lemma[1]

Let $A \in (\mathcal{M}_D^{sa})^g$, $B \in (\mathcal{M}_d^{sa})^g$. Furthermore, let $\mathcal{D}_A(1)$ be bounded. The unital linear map $\Phi : \mathrm{span}\{I, A_1, \ldots, A_g\} \to \mathcal{M}_d^{sa}$,

$$\Phi : A_i \mapsto B_i \qquad \forall i \in [g]$$

is n-positive if and only if $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$.

- $\mathcal{D}_A(1) \subseteq \mathcal{D}_B(1) \implies s \cdot \mathcal{D}_A \subseteq \mathcal{D}_B$ for $s \in [0,1]^g$.
- Inclusion set: $\Delta(g,d) := \Big\{ s \in [0,1]^g : \forall B \in \left(\mathcal{M}_d^{sa}\right)^g$ $\mathcal{D}_{\diamond,g}(1) \subseteq \mathcal{D}_B(1) \Rightarrow s \cdot \mathcal{D}_{\diamond,g} \subseteq \mathcal{D}_B \Big\}$

[1] J. W. Helton et al. Dilations, linear matrix inequalities, the matrix cube problem and beta distributions. *Memoirs of the AMS*, 275(1232), 2019.

Connecting the two problems

Theorem

Let $E \in \left(\mathcal{M}_d^{sa}\right)^g$ and let $2E - I := (2E_1 - I_d, \ldots, 2E_g - I_d)$. We have

1. $\mathcal{D}_{\diamond,g}(1) \subseteq \mathcal{D}_{2E-I}(1)$ *if and only if* $E_1, \ldots, E_g$ *are POVM elements.*
2. $\mathcal{D}_{\diamond,g} \subseteq \mathcal{D}_{2E-I}$ *if and only if* $E_1, \ldots, E_g$ *are jointly measurable POVM elements.*
3. $\mathcal{D}_{\diamond,g}(k) \subseteq \mathcal{D}_{2E-I}(k)$ *for* $k \in [d]$ *if and only if for any isometry* $V : \mathbb{C}^k \hookrightarrow \mathbb{C}^d$*, the induced compressions* $V^* E_1 V, \ldots, V^* E_g V$ *are jointly measurable POVM elements.*

Theorem

It holds that $\Gamma(g, d) = \Delta(g, d)$.

Proof ideas:

$\mathcal{D}_{\diamond,g}(1) \subseteq \mathcal{D}_{2E-I}(1)$ if and only if $E_1, \dots, E_g$ are POVM elements.

- Consider the extreme points $\pm e_i$ of the matrix diamond.

$\mathcal{D}_{\diamond,g} \subseteq \mathcal{D}_{2E-I}$ if and only if $E_1, \dots, E_g$ are jointly measurable POVM elements.

- Inclusion holds if and only if the unital map

$$\Phi : I_2^{\otimes(i-1)} \otimes \mathrm{diag}[-1, 1] \otimes I_2^{\otimes(g-i)} \mapsto 2E_i - I_d$$

  is completely positive
- Arveson's extension theorem: Φ has a positive extension $\tilde{\Phi}$ to $\mathbb{C}^{2^g}$
- Basis g_η of $\mathbb{C}^{2^g}$: $G_\eta := \tilde{\Phi}(g_\eta)$ is a joint POVM for $E_1, \dots, E_g$ if and only if $\tilde{\Phi}$ positive

Points in the inclusion set

It holds that $\Gamma(g, d) = \Delta(g, d)$.

- Davidson et al.[2]: Point independent of d

$$\frac{1}{g}(1, \ldots, 1) \in \Delta(g, d)$$

- Helton et al.[3]: Point independent of g

$$\frac{1}{2d}(1, \ldots, 1) \in \Delta(g, d)$$

[2] K. R. Davidson et al. Dilations, inclusions of matrix convex sets, and completely positive maps. *Int. Math. Res. Notices*, 2017(13):4069–4130, 2017.

[3] J. W. Helton et al. Dilations, linear matrix inequalities, the matrix cube problem and beta distributions. *Memoirs of the AMS*, 275(1232), 2019.

Upper and lower bounds for the matrix diamond [4]

Theorem

Let $g, d \in \mathbb{N}$. Then, it holds that $\mathrm{QC}_g \subseteq \Delta(g,d)$. In other words, for any g-tuple $E_1, \ldots, E_g$ of POVM elements and any positive vector $s \in \mathbb{R}_+^g$ with $\|s\|_2 \leq 1$, the g-tuple of noisy POVM elements $E_i' = s_i E_i + (1 - s_i) I_d/2$ is jointly measurable.

Theorem

Let $g \geq 2$, $d \geq 2^{\lceil (g-1)/2 \rceil}$. Then, $\Delta(g,d) \subseteq \mathrm{QC}_g$.

$$\mathrm{QC}_g := \{s \in [0,1]^g : s_1^2 + \ldots + s_g^2 \leq 1\}$$

[4] B. Passer et al. Minimal and maximal matrix convex sets. *J. Funct. Anal.* , 274:3197–3253, 2018.

Maximally incompatible measurements

We can construct POVM elements which achieve the upper bound:

$$F_i^{(k+1)} = \sigma_X \otimes F_i^{(k)} \qquad \forall i \in [2k+1]$$

$$F_{2k+2}^{(k+1)} = \sigma_Y \otimes I_{2^k}, \qquad F_{2k+3}^{(k+1)} = \sigma_Z \otimes I_{2^k}.$$

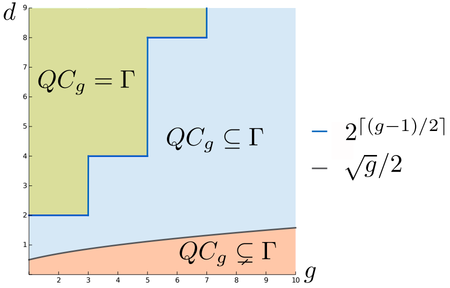
Example

$k = 1$: $F_1^{(1)} = \sigma_X$, $F_2^{(1)} = \sigma_Y$, $F_3^{(1)} = \sigma_Z$
$k = 2$:

$$F_1^{(2)} = \sigma_X \otimes \sigma_X, \qquad F_2^{(2)} = \sigma_X \otimes \sigma_Y, \qquad F_3^{(2)} = \sigma_X \otimes \sigma_Z,$$

$$F_4^{(2)} = \sigma_Y \otimes I_2, \qquad F_5^{(2)} = \sigma_Z \otimes I_2$$

What we know about $\Gamma(g, d)$



- In the green area, the upper and lower bound from Passer et al. coincide
- In the orange area, we know that the point $1/(2d)(1, \ldots, 1)$ is no longer in QC_g
- Lower bounds better than symmetric cloning
- Attention: QC_g shrinks with g

Outlook: More outcomes

The matrix diamond is the universal for binary measurements, which object do we consider for more outcomes?

- Line with endpoints ± 1 is a simplex S_1 in one dimension
- $\mathcal{D}_{\diamond,2}(1) = S_1 \oplus S_1$
- Measurements with k-outcomes: S_{k-1}
- Level 1: $S_{k_1-1} \oplus \ldots \oplus S_{k_g-1}$
- Matrix diamond is the maximal free spectrahedron sitting on the ℓ_1-ball
- Taking the maximal free spectrahedron for k-outcomes leads to the matrix jewel
- Connection carries over to the general setting
- Similar inclusion problems can be found for the compatibility of quantum channels and compatibility in GPTs (ongoing)

Conclusion

- Compatibility of binary POVMs corresponds to inclusion of the matrix diamond into a free spectrahedron defined by the POVM elements
- Compatibility region = Inclusion set of the matrix diamond
- $\Gamma(g, d) = \mathrm{QC}_g$ for dimension d exponential in the number of measurements g

References:

1. AB and Ion Nechita. Joint measurability of quantum effects and the matrix diamond. *Journal of Mathematical Physics*, 59(11):112202, 2018.
2. AB and Ion Nechita. Compatibility of quantum measurements and inclusion constants for the matrix jewel. *arXiv1809.04514*, 2018.