Free spectrahedra in quantum information theory

Andreas Bluhm
Univ. Grenoble Alpes, CNRS, Grenoble INP, LIG

June 12, 2023, ILAS 2023 Madrid

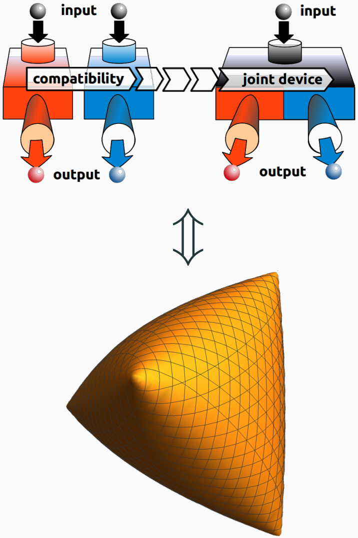
Talk outline

Measurement compatibility

Free spectrahedra

Connecting the two

Polytope compatibility



joint work with I. Nechita and S. Schmidt

Measurement compatibility

Quantum states and measurements

- Motivation: Classical state $\rightsquigarrow$ probability distributions: $p \in \mathbb{R}^d$, $p \geq 0$, $\sum_i p_i = 1$.
- Quantum states $\rightsquigarrow$ density matrices: $\rho \in \mathcal{M}_d(\mathbb{C})$, $\rho \geq 0$, $\operatorname{Tr} \rho = 1$.
- Measurement outcomes are labeled $\{1, \ldots, k\}$, need to be assigned probabilities.
- Measurements: Tuples of matrices $(E_1, \ldots, E_k)$ such that $(\operatorname{Tr}[E_1 \rho], \ldots, \operatorname{Tr}[E_k \rho])$ is a probability distribution for all states ρ.
  - $\operatorname{Tr}[E_i \rho] \in \mathbb{R} \rightsquigarrow E_i = E_i^*$.
  - $\operatorname{Tr}[E_i \rho] \geq 0 \rightsquigarrow E_i \geq 0$.
  - $\sum_i \operatorname{Tr}[E_i \rho] = 1 \rightsquigarrow \sum_i E_i = I_d$.
- Tuples of PSD matrices summing to identity are called positive operator-valued measures (POVMs).

Quantum measurements: Compatibility

- Quantum measurements $\rightsquigarrow$ give the probabilities of the classical outcomes when a quantum state enters a measurement apparatus. Mathematically, measurements are modeled by POVMs.

Definition

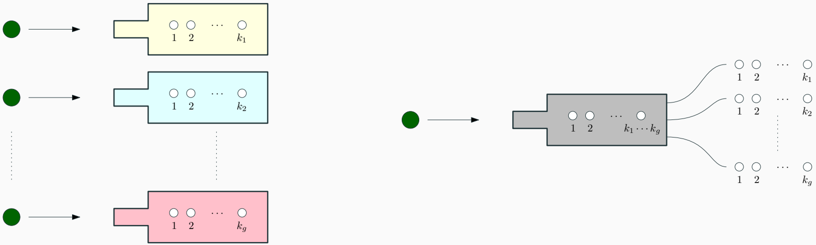
Two POVMs, $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_l)$, are called compatible if there exists a third POVM $C = (C_{ij})_{i \in [k], j \in [l]}$ such that

$$\forall i \in [k], \quad A_i = \sum_{j=1}^{l} C_{ij} \qquad \text{and} \qquad \forall j \in [l], \quad B_j = \sum_{i=1}^{k} C_{ij}.$$

The definition generalizes to g-tuples of POVMs $A^{(1)}, \ldots, A^{(g)}$, having respectively $k_1, \ldots k_g$ outcomes, where the joint POVM C has outcome set $[k_1] \times \cdots \times [k_g]$.

- Other way to say that: jointly measurable.

What does it mean?



- Compatible measurements can be simulated by a single joint measurement, by classically post-processing its outputs $A_i^{(j)} = \sum_\lambda p_j(i|\lambda) C_\lambda$.
- Examples:
  1. Trivial POVMs $A = (p_i I_d)$ and $B = (q_j I_d)$ are compatible.
  2. Commuting POVMs $[A_i, B_j] = 0$ are compatible.
  3. If the POVM A is projective, then A and B are compatible if and only if they commute.

Noisy POVMs

- POVMs can be made compatible by adding noise, i.e. mixing in trivial POVMs.
- Example: dichotomic POVMs and white noise, $s \in [0,1]$:

$$(E, I-E) \mapsto s(E, I-E) + (1-s)(\frac{I}{2}, \frac{I}{2}) \qquad \text{or} \qquad E \mapsto sE + (1-s)\frac{I}{2}.$$

- Taking $s = 1/2$ suffices to render any pair of dichotomic POVMs compatible $\rightsquigarrow$ define $C_{ij} := (A_i + B_j)/4$.
- From now on, we focus on dichotomic (YES/NO) POVMs.

Definition

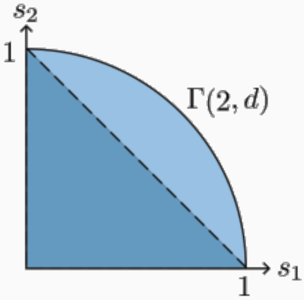
The compatibility region for g measurements on $\mathbb{C}^d$ is the set

$$\Gamma(g,d) := \{s \in [0,1]^g \,:\, \text{for all quantum effects } E_1, \ldots, E_g \in \mathcal{M}_d(\mathbb{C}), \text{ the noisy versions } s_i E_i + (1-s_i) I_d/2 \text{ are compatible}\}$$

Compatibility region

$$\Gamma(g,d) := \{s \in [0,1]^g \ : \text{ for all quantum effects } E_1,\ldots,E_g \in \mathcal{M}_d(\mathbb{C}), \text{ the noisy versions } s_iE_i + (1-s_i)I_d/2 \text{ are compatible}\}$$

- The set $\Gamma(g,d)$ is convex.
- For all $i \in [g]$, $e_i \in \Gamma(g,d)$: every measurement is compatible with $g-1$ trivial measurements.
- For $d \geq 2$, $(1,1,\ldots,1) \notin \Gamma(g,d)$: there exist incompatible measurements.
- For all $d \geq 2$, $\Gamma(2,d)$ is a quarter-circle.



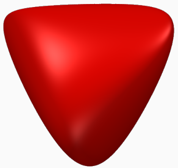
Generally speaking, the set $\Gamma(g,d)$ tells us how robust (to noise) is the incompatibility of g dichotomic measurements on $\mathbb{C}^d$.

Free spectrahedra

Free spectrahedra

- A spectrahedron is given by PSD constraints: for $A = (A_1, \ldots, A_g) \in (\mathcal{M}_d^{\mathrm{sa}}(\mathbb{C}))^g$

$$\mathcal{D}_A(1) := \left\{ x \in \mathbb{R}^g \ : \ \sum_{i=1}^{g} x_i A_i \leq I_d \right\}$$



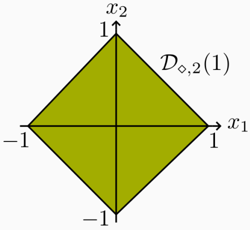
- $\mathcal{D}_{(\sigma_X, \sigma_Y, \sigma_Z)}(1) = \{(x, y, z) \in \mathbb{R}^3 \ : \ x\sigma_X + y\sigma_Y + z\sigma_Z \leq I_2\} = \text{Bloch ball}$
- A free spectrahedron is the matricization of a spectrahedron

$$\mathcal{D}_A := \bigsqcup_{n=1}^{\infty} \mathcal{D}_A(n) \quad \text{with} \quad \mathcal{D}_A(n) := \left\{ X \in (\mathcal{M}_n^{\mathrm{sa}}(\mathbb{C}))^g \ : \ \sum_{i=1}^{g} X_i \otimes A_i \leq I_{nd} \right\}$$

Example: the matrix diamond

The matrix diamond is the free spectrahedron defined by

$$\mathcal{D}_{\diamondsuit,g} := \bigsqcup_{n=1}^{\infty}\{X \in (\mathcal{M}_n^{\mathrm{sa}}(\mathbb{C}))^g : \sum_{i=1}^{g} \epsilon_i X_i \leq I_n, \quad \forall \epsilon \in \{\pm 1\}^g\}$$



- At level one, $\mathcal{D}_{\diamondsuit,g}(1)$ is the unit ball of the ℓ_1 norm on $\mathbb{R}^g$
- As a free spectrahedron, it is defined by $2^g \times 2^g$ diagonal matrices $\mathcal{D}_{\diamondsuit,g} = \mathcal{D}_{L_1,\ldots,L_g}$, with $L_i = I_2 \otimes \cdots \otimes I_2 \otimes \mathrm{diag}(1,-1) \otimes I_2 \otimes \cdots \otimes I_2$

Spectrahedral inclusion

- Consider two free spectrahedra defined by $(A_1, \dots, A_g)$ and $(B_1, \dots, B_g)$
- We write $\mathcal{D}_A \subseteq \mathcal{D}_B$ if, for all $n \geq 1$, $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$
- Clearly, $\mathcal{D}_A \subseteq \mathcal{D}_B \implies \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$. For the converse implication to hold, one may need to shrink $\mathcal{D}_A$...

Definition

For a free spectrahedron $\mathcal{D}_A$, we define its set of inclusion constants as

$$\Delta_A(g, d) := \{s \in [0,1]^g \ : \text{for all } g\text{-tuples } B_1, \dots, B_g \in \mathcal{M}_d(\mathbb{C})^{\mathrm{sa}}, \\ \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1) \implies s.\mathcal{D}_A \subseteq \mathcal{D}_B\}$$

- The inclusion constants for the matrix cube play an important role in combinatorial optimization
- We shall be concerned with the inclusion set for the matrix diamond, which we denote by $\Delta(g, d)$

Connecting the two

Compatibility in QM $\iff$ matrix diamond inclusion

To a g-tuple $E \in (\mathcal{M}_d^{\mathrm{sa}}(\mathbb{C}))^g$, we associate:

$$\mathcal{D}_{2E-I} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{\mathrm{sa}}(\mathbb{C}))^g \, : \, \sum_{i=1}^{g} X_i \otimes (2E_i - I_d) \leq I_{nd}\}$$

Theorem

Let $E \in (\mathcal{M}_d^{sa}(\mathbb{C}))^g$ be g-tuple of selfadjoint matrices. Then:

- *The matrices E are quantum effects $\iff \mathcal{D}_{\diamondsuit,g}(1) \subseteq \mathcal{D}_{2E-I}(1)$*
- *The matrices E are compatible quantum effects $\iff \mathcal{D}_{\diamondsuit,g} \subseteq \mathcal{D}_{2E-I}$*

At the intermediate levels $1 \leq n \leq d$, $\mathcal{D}_{\diamondsuit,g}(n) \subseteq \mathcal{D}_{2E-I}(n)$ iff for all isometries $V : \mathbb{C}^n \to \mathbb{C}^d$, the compressed effects $V^* E_i V$ are compatible.

Moreover, the compatibility region is equal to the set of inclusion constants of the matrix diamond: $\forall g, d$, $\Gamma(g,d) = \Delta(g,d)$.

Consequences

Many things are known about the matrix diamond

- For all g, d, $\frac{1}{2d}(1, 1, \ldots, 1) \in \Delta(g, d)$ (Helton *et al.*, 2019)
- For all g, d, $\mathrm{QC}_g := \{s \in [0,1]^g \,:\, \sum_i s_i^2 \leq 1\} \subseteq \Delta(g, d)$ (Passer *et al.*, 2018)

Many things are known about (in-)compatibility

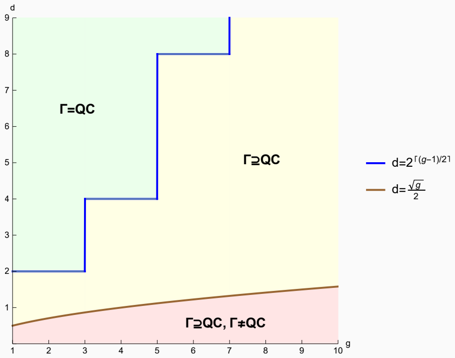
- Some small g, d cases completely solved
- Approximate quantum cloning $\implies$ compatibility

$$\mathrm{Clone}(g,d) := \{s \in [0,1]^g \,:\exists \text{ quantum channel } \Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})^{\otimes g} \text{ s.t.}$$
$$\forall i \in [g], \quad \Phi_i(X) = s_i X + (1 - s_i)\frac{\operatorname{Tr} X}{d} I_d\}$$

Theorem

For all g and $d \geq 2^{\lceil (g-1)/2 \rceil}$, $\Gamma(g,d) = \Delta(g,d) = \mathrm{QC}_g$

Phase diagram



- Connection to free spectrahedra also holds for arbitrary outcomes
- Instead of matrix diamond, consider its generalization, the matrix jewel
- We can get better lower bounds from the matrix cube (duality, can be seen as steering)

Inclusion of spectrahedra and (completely) positive maps

Theorem (Helton et al., 2013)

Let $A \in (\mathcal{M}_D^{sa}(\mathbb{C}))^g$, $B \in (\mathcal{M}_d^{sa}(\mathbb{C}))^g$ such that $\mathcal{D}_A(1)$ is bounded. Then, $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$ iff the unital linear map

$$\Phi : \mathrm{span}\{I, A_1, \ldots, A_g\} \to \mathcal{M}_d^{sa}(\mathbb{C}), \qquad A_i \mapsto B_i$$

is n-positive.

Sketch of the proof of the main theorem:

- Level 1: the extremal points of $\mathcal{D}_{\diamondsuit,g}(1)$ are $\pm e_i$
- The inclusion $\mathcal{D}_{\diamondsuit,g} \subseteq \mathcal{D}_{2E-I}$ holds iff the unital map $\Phi : I_2 \otimes \cdots \otimes I_2 \otimes \mathrm{diag}(1,-1) \otimes I_2 \otimes \cdots \otimes I_2 \mapsto 2E_i - I_d$ is CP
- Arveson's extension theorem: Φ has a (completely) positive extension $\tilde{\Phi}$ to $\mathbb{R}^{2^g}$
- $C_f := \tilde{\Phi}(f)$ is a joint POVM for the E_i's, where $\{f\}$ is a basis of $\mathbb{R}^{2^g}$

Polytope compatibility

Polytope compatibility

- We have seen that measurement compatibility can be reformulated as the inclusion of a free spectrahedron corresponding to the ℓ_1 unit ball.
- What happens if we replace this unit ball by some other polytope?
- Which quantum information problems can be described using these tools?

Magic squares

A magic square is a collection of positive operators A_{ij}, $i, j \in [N]$, such that

$$\begin{array}{ccccccccc}
A_{11} & + & A_{12} & + & \dots & + & A_{1N} & = I \\
+ & & + & & & & + & \\
\vdots & & \vdots & & \ddots & & \vdots & \vdots \\
+ & & + & & & & + & \\
A_{N1} & + & A_{N2} & + & \dots & + & A_{NN} & = I \\
\| & & \| & & & & \| & \\
I & & I & & \dots & & I &
\end{array}$$

The magic square is said to be semiclassical if

$$A = \sum_{i,j \in [N]} |i\rangle\langle j| \otimes A_{ij} = \sum_{\pi \in \mathcal{S}_N} P_\pi \otimes Q_\pi,$$

where P_π is the permutation matrix associated to π and $\{Q_\pi\}_\pi$ is a POVM.

Birkhoff polytope compatibility

Definition

For a given $N \geq 2$, the Birkhoff body $\mathcal{B}_N(1)$ is defined as the set of $(N-1) \times (N-1)$ truncations of $N \times N$ bistochastic matrices, shifted by J/N:

$$\mathcal{B}_N = \{A^{(N-1)} - J_{N-1}/N \,:\, A \in \mathcal{M}_N(\mathbb{R}) \text{ bistochastic}\} \subset \mathbb{R}^{(N-1)^2}.$$

Theorem

Consider a $(N-1)^2$-tuple of selfadjoint matrices $A \in \mathcal{M}_d^{sa}(\mathbb{C})^{(N-1)^2}$ and a corresponding matrix $\tilde{A} \in \mathcal{M}_N(\mathcal{M}_d(\mathbb{C}))$. Then:

1. *The matrix $\tilde{A}$ is a magic square if and only if $\mathcal{D}_{\mathcal{B}_N^\circ}(1) \subseteq \mathcal{D}_{A-I/N}(1)$ (the $A - I/N$ are $\mathcal{B}_N$-operators).*
2. *The matrix $\tilde{A}$ is a semiclassical magic square if and only if $\mathcal{D}_{\mathcal{B}_N^\circ} \subseteq \mathcal{D}_{A-I/N}$ (the $A - I/N$ are $\mathcal{B}_N$-compatible).*

Relation to measurement incompatibility

Is being a semiclassical magic square the same as being compatible? No.

$\frac{1}{2}\lvert 0\rangle\langle 0\rvert$	$\frac{1}{2}\lvert 1\rangle\langle 1\rvert$	0	$\frac{1}{2}I_2$
$\frac{1}{2}\lvert 1\rangle\langle 1\rvert$	$\frac{1}{2}\lvert 0\rangle\langle 0\rvert$	$\frac{1}{2}I_2$	0
0	$\frac{1}{2}I_2$	$\frac{1}{2}\lvert +\rangle\langle +\rvert$	$\frac{1}{2}\lvert -\rangle\langle -\rvert$
$\frac{1}{2}I_2$	0	$\frac{1}{2}\lvert -\rangle\langle -\rvert$	$\frac{1}{2}\lvert +\rangle\langle +\rvert$

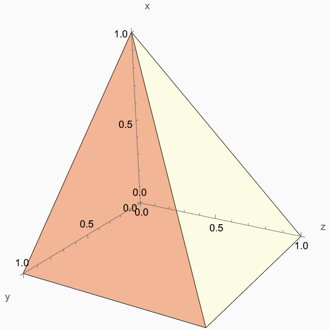
These measurements are compatible, but they do not form a semiclassical magic square.

Reason: $\mathcal{B}_N$-compatibility restricts the post-processing to $p_i(j|\lambda) = p_j(i|\lambda)$, i.e., enforces special structure in the joint POVM.

Measurement compatibility with shared effects

Can we generalize the magic square example?

$$\mathcal{P} = (-1/3, -1/3, -1/3) + \mathrm{conv}\{((1,0,0),(0,0,0),(0,0,1),(0,1,0),(0,1,1)\}.$$



Consider $(A, B, C) \in (\mathcal{M}_d^{\mathrm{sa}}(\mathbb{C}))^3$. Then, we have $(A, B, C) + 1/3(I, I, I)$ are $\mathcal{P}$-operators if and only if both $(A, B, I_d - A - B)$ and $(A, C, I_d - A - C)$ are POVMs.

Measurement compatibility with shared effects, continued

When does $(A, B, C) + 1/3(I, I, I)$ $\mathcal{P}$-compatible hold? Equivalent to the existence of a joint measurement such that

Q_1	0	0	$= A$
0	Q_5	Q_4	$= B$
0	Q_3	Q_2	$= I_d - A - B$
$= A$	$= C$	$= I_d - A - C$	

Not all joint measurements are of this form, check

$$\left(\frac{1}{2} I_2, \frac{1}{2}\left|0\middle\rangle\middle\langle 0\right|, \frac{1}{2}\left|1\middle\rangle\middle\langle 1\right|\right) \qquad \text{and} \qquad \left(\frac{1}{2} I_2, \frac{1}{2}\left|+\middle\rangle\middle\langle +\right|, \frac{1}{2}\left|-\middle\rangle\middle\langle -\right|\right).$$

$\mathcal{P}$-compatibility for general polytopes $\mathcal{P}$ corresponds to measurement compatibility with shared elements and restricted post-processing, according to a graph defined by $\mathcal{P}$.

Summary

- Measurement incompatibility can be phrased as inclusion of free spectrahedra. Base set: diamond.
- Noise robustness corresponds to inclusion constants.
- Generalization: $\mathcal{P}$-compatible operators.
- Examples include magic squares and compatibility with shared elements (under restricted post-processing)

References

Inclusion constants:

[1] J. W. Helton, I. Klep, S. A. McCullough, M. Schweighofer: *Dilations, linear matrix inequalities, the matrix cube problem and beta distributions*. Mem. Amer. Math. Soc. 257 vol. 1232, 2019

[2] B. Passer, O. Shalit, B. Solel: *Minimal and maximal matrix convex sets*. J. Funct. Anal. 274(11), 2018

Compatibility and free spectrahedra:

[3] AB, I. Nechita: *Joint measurability of quantum effects and the matrix diamond*. Journal of Mathematical Physics, 59, 2018

[4] AB, I. Nechita: *Compatibility of Quantum Measurements and Inclusion Constants for the Matrix Jewel*. SIAM Journal on Applied Algebra and Geometry, 4(2), 2018

[5] AB, I. Nechita, S. Schmidt: *Polytope compatibility – from quantum measurements to magic squares*. arXiv-preprint arxiv:2304.10920