Compatibility of quantum effects and inclusion of free spectrahedra

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Compatibility: Two binary measurements

Measurement = POVM: $E_i \geq 0$, $\sum_{i=1}^{k} E_i = I$

Example

Consider two binary measurements: $\{E, I - E\}$, $\{F, I - F\}$.
Assume that there is a measurement $\{R_{i,j}\}_{i,j=0}^{1}$ such that

$$
\begin{array}{ccccc}
R_{0,0} & + & R_{0,1} & = & E \\
+ & & + & & \\
R_{1,0} & + & R_{1,1} & = & I - E \\
\| & & \| & & \\
F & & I - F & &
\end{array}
$$

Then the measurements are jointly measurable or compatible.

The compatibility region

- Measurements can be made compatible by adding a sufficient amount of noise
- White noise:

$$E \mapsto sE + \frac{1-s}{2} I_d, \qquad s \in [0,1]$$

- Compatibility region:

$$\Gamma(g,d) := \left\{ s \in [0,1]^g : \; s_i E_i + \frac{1-s_i}{2} I_d \text{ are compatible } \forall E_1, \ldots, E_g \in \mathrm{Eff}_d \right\}.$$

- Incompatibility is a resource for quantum information processing
- Noise robustness can be used to quantify incompatibility

Free spectrahedra

Let $A \in (M_d^{sa})^g$. The free spectrahedron at level n is defined as

$$\mathcal{D}_A(n) := \left\{ X \in \left(\mathcal{M}_n^{sa}\right)^g : \sum_{i=1}^{g} A_i \otimes X_i \leq I_{nd} \right\}.$$

The free spectrahedron is the union of these levels

$$\mathcal{D}_A := \bigcup_{n \in \mathbb{N}} \mathcal{D}_A(n).$$

An important example is the matrix diamond:

$$\mathcal{D}_{\diamond,g}(n) = \left\{ X \in \left(\mathcal{M}_n^{sa}\right)^g : \sum_{i=1}^{g} \epsilon_i X_i \leq I_n \ \forall \epsilon \in \{-1,+1\}^g \right\}.$$

Inclusion of free spectrahedra

- $\mathcal{D}_A \subseteq \mathcal{D}_B$ means $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$ for all n
- $\mathcal{D}_A(1) \subseteq \mathcal{D}_B(1) \implies s \cdot \mathcal{D}_A \subseteq \mathcal{D}_B$ for $s \in [0,1]^g$.
- Inclusion set:

$$\Delta(g,d) := \Big\{ s \in [0,1]^g : \forall B \in \left(\mathcal{M}_d^{sa}\right)^g \\ \mathcal{D}_{\diamond,g}(1) \subseteq \mathcal{D}_B(1) \Rightarrow s \cdot \mathcal{D}_{\diamond,g} \subseteq \mathcal{D}_B \Big\}$$

Connecting the two problems

Theorem

Let $E \in \left(\mathcal{M}_d^{sa}\right)^g$ *and let* $2E - I := (2E_1 - I_d, \dots, 2E_g - I_d)$*. We have*

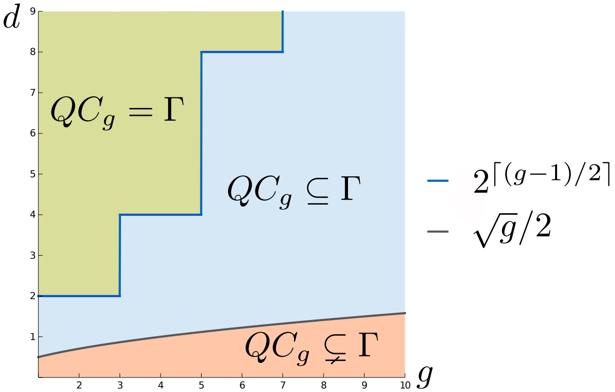
1. $\mathcal{D}_{\diamond,g}(1) \subseteq \mathcal{D}_{2E-I}(1)$ *if and only if* $E_1, \dots, E_g$ *are POVM elements.*
2. $\mathcal{D}_{\diamond,g} \subseteq \mathcal{D}_{2E-I}$ *if and only if* $E_1, \dots, E_g$ *are jointly measurable POVM elements.*
3. $\mathcal{D}_{\diamond,g}(k) \subseteq \mathcal{D}_{2E-I}(k)$ *for* $k \in [d]$ *if and only if for any isometry* $V : \mathbb{C}^k \hookrightarrow \mathbb{C}^d$*, the induced compressions* $V^* E_1 V, \dots, V^* E_g V$ *are jointly measurable POVM elements.*

Theorem

It holds that $\Gamma(g, d) = \Delta(g, d)$*.*

What we know about $\Gamma(g, d)$

$$QC_g := \{s \in [0,1]^g : s_1^2 + \ldots + s_g^2 \leq 1\}$$



- In the orange area, we know that the point $1/(2d)(1, \ldots, 1) \in \Gamma(g, d)$ is no longer in QC_g

Conclusion

- Compatibility of binary POVMs corresponds to inclusion of the matrix diamond into a free spectrahedron defined by the POVM elements
- Compatibility region = Inclusion set of the matrix diamond
- $\Gamma(g, d) = QC_g$ for dimension d exponential in the number of measurements g

References:

1. AB and Ion Nechita. Joint measurability of quantum effects and the matrix diamond. *Journal of Mathematical Physics*, 59(11):112202, 2018.
2. AB and Ion Nechita. Compatibility of quantum measurements and inclusion constants for the matrix jewel. *arXiv1809.04514*, 2018.