# Quantum compression relative to a set of measurements

Andreas Bluhm

(with Michael M. Wolf and Lukas Rauber)

Technical University of Munich
Department of Mathematics

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# Measurements

Aim of this talk: Compression of Hilbert space dimension

Measurement upon preparation $\rho \in \mathcal{S}(\mathbb{C}^D)$:

- Measurement outcomes $\{a_i\}_{i=1}^k$, probabilities $\{p_i\}_{i=1}^k$
- Effect operators:

$$\mathcal{E}(\mathbb{C}^D) = \{ E \in \mathcal{M}_D : 0 \leq E \leq \mathbb{1} \}$$

- Associate with probability:

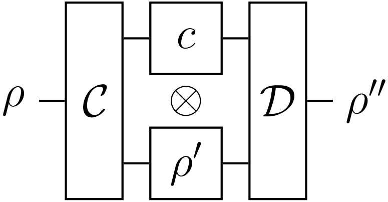
$$p_i = \mathrm{Tr}\,(E_i \rho) \qquad \forall i \in \{ 1, \ldots, k \}$$

- Normalization:

$$\mathbb{1} = \sum_{i=1}^{k} E_i$$

From now on: Measurement = Set of effect operators

# Setup




$$\mathcal{M}_D \qquad \mathcal{M}_d \otimes \mathbb{C}^n \qquad \mathcal{M}_D$$

- $\mathcal{C}$, $\mathcal{D}$ completely positive trace preserving maps, $\mathcal{T} = \mathcal{D} \circ \mathcal{C}$
- Fixed set $\mathcal{O} \subset \mathcal{E}(\mathbb{C}^D)$ of effect operators
- Constraints:

$$\mathrm{Tr}\,(\rho E) = \mathrm{Tr}\,(\mathcal{T}(\rho) E) \qquad \forall E \in \mathcal{O}$$

- Require this to hold for all $\rho \in \mathcal{S}(\mathbb{C}^D)$
- Aim: Find $d$ as small as possible, $n$ may be arbitrarily large
- This $d$ is the compression dimension of $\mathcal{O}$

# Stability of the compression dimension

## Theorem

*Let $d$ be the compression dimension for $\mathcal{O}$ and $\mathcal{O}$ be compact. Then there is an $\epsilon > 0$ such that for $d' < d$ and for every CPTP maps $\mathcal{C} : \mathcal{M}_D \to \mathcal{M}_{d'} \otimes \mathbb{C}^n$, $\mathcal{D} : \mathcal{M}_{d'} \otimes \mathbb{C}^n \to \mathcal{M}_D$, $n \in \mathbb{N}$, there are $\rho \in \mathcal{S}(\mathbb{C}^D)$ and $E \in \mathcal{O}$ such that*

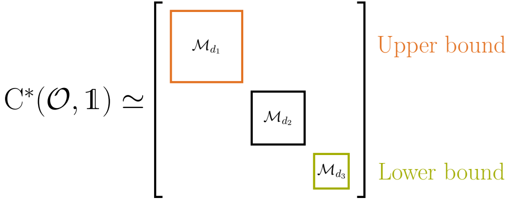
$$|\mathrm{Tr}\,(\rho E) - \mathrm{Tr}\,((\mathcal{D} \circ \mathcal{C})[\rho]E)\,| \geq \epsilon.$$

Proof idea:

- Classical side information can be bounded, i.e. for every $\mathcal{T} = \mathcal{D} \circ \mathcal{C}$, there are $\mathcal{C}'$, $\mathcal{D}'$ with $n \leq D^4$ and $\mathcal{T} = \mathcal{D}' \circ \mathcal{C}'$.
- Then compactness argument

For the rest of the talk, we can restrict to the exact case.

# Algebraic approach




## Theorem (Lower/Upper bound compression dimension)

*Let $d$ be the compression dimension of $\mathcal{O}$ and $\{ D_i \}$ be the dimensions of the matrix algebras occuring in the representation of $\mathrm{C}^*(\mathcal{O}, \mathbb{1})$. Then it holds that*

$$\min_{i\in[s]} D_i \leq d \leq \max_{i\in[s]} D_i .$$

# Examples

## Example (Generic case)

Let $A$, $B \in \mathcal{M}_D^{\mathrm{herm}}$ be generic. Then $\mathrm{C}^*(\{A, B\}) \simeq \mathcal{M}_D$ and $\mathcal{O} := \{\mathbb{1}, A, B\}$ is incompressible ($d = D$).

## Example (Two bipartite von Neumann measurements)

Let $P$, $Q \in \mathcal{E}(\mathbb{C}^D)$ be two orthogonal projections. Then, $\mathcal{O} := \{P, \mathbb{1} - P, Q, \mathbb{1} - Q\}$ has compression dimension at most 2.

The latter example breaks down for more projections.

# Computing the dimension

What can we say if for $\mathrm{C}^*(\mathcal{O}) \simeq \bigoplus_{i=1}^{s} \mathcal{M}_{D_i}$ upper and lower bound do not match?

- Can phrase this as an interpolation problem: Check if there is a CP map $\Phi : \mathcal{M}_D \to \mathcal{M}_{D_1}$ such that

$$E^1 = \Phi\left(\begin{bmatrix} 0 & & & \\ & E^2 & & \\ & & \ddots & \\ & & & E^s \end{bmatrix}\right) \qquad \forall E \in \mathcal{O}.$$

- This can be checked using a semidefinite program[1]
- Repeat for every block from largest to smallest until interpolation map no longer exists

[1] T. Heinosaari, M. A. Jivulescu, D. Reeb and M. M. Wolf. Extending quantum operations. *Journal of Mathematical Physics, 53(10):102208, 2012.*

# Geometric approach

Irreducible factors:

$$p(x, z) = \prod_{i=1}^{s} p_i^{n_i}$$

Here, $p_i$ is a polynomial of degree $D_i$.

## Theorem (Geometric lower bounds)

*Let $\mathcal{O} \subset \mathcal{E}(\mathbb{C}^D)$ and $E_1, E_2 \in \mathrm{span}_{\mathbb{R}}\{\mathcal{O}, \mathbb{1}\}$. Further, let $p(x, z) := \det[x\mathbb{1} - E_1 - zE_2]$. Then the smallest among the degrees of the irreducible factors of p is a lower bound on the compression dimension of $\mathcal{O}$. In particular, if p is irreducible over the reals, then $\mathcal{O}$ is incompressible.*

- Corollary: The same holds if we allow multiple copies $\rho^{\otimes m}$ instead of $\rho$ as input for $\mathcal{C}$.
- This cannot be proved using the algebraic approach.

# Proof techniques

Constraints:

$$\mathrm{Tr}\,(\rho E) = \mathrm{Tr}\,(\mathcal{T}(\rho)E) = \mathrm{Tr}\,(\rho\mathcal{T}^*(E)) \qquad \forall \rho \in \mathcal{S}(\mathbb{C}^D), \forall E \in \mathcal{O}$$

Algebraic:

- Use result by Arveson[2] about the fixed points of a unital completely positive map $\mathcal{T}$.
- If those algebraically generate $\mathcal{M}_D$, then $\mathcal{T} = \mathrm{id}$.

Geometric:

- Reduce problem to determining whether for $E_1, E_2 \in \mathcal{O}$

$$\|E_1 + tE_2\|_\infty = \|\mathcal{D}^*(E_1) + t\mathcal{D}^*(E_2)\|_\infty \qquad \forall t \in \mathbb{R}$$

- Bézout's theorem: Not possible if $E_1$, $E_2$ give rise to an irreducible characteristic polynomial

[2] William Arveson: Subalgebras of C$^*$-algebras II. *Acta Mathematica*, 128(1):271–308, 1972.

# Conclusion

- Generically no compression possible with respect to Hilbert space dimension
- Physical setups: Dimension of largest block achievable, dimension of smallest block lower bound
- Compression dimension can be checked algorithmically
- Algebraic argument stronger statement, geometric more widely applicable

Reference: A. Bluhm, L. Rauber and M. M. Wolf. Quantum compression relative to a set of measurements. *Annales Henri Poincaré*, 19(6):1891–1937, 2018.