Maximal violation of steering inequalities and the matrix cube

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Talk outline

- Main topic: Maximal violation of steering inequalities

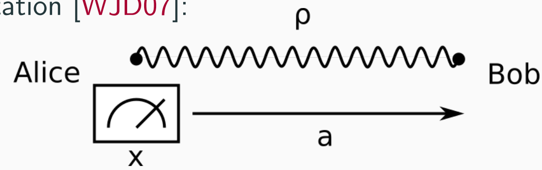
Question:
For an arbitrary steering inequality, how much smaller is its LHS value than its quantum value?

- Main tool: Connection to free spectrahedra, in particular to the matrix cube

Quantum steering

Local hidden state models

Operational interpretation [WJD07]:



- g measurements on Alice's side, k_x outcomes each, quantum systems of dimension d
- Assemblage: tuple $(\sigma_{a|x})_{a,x}$ of positive matrices such that

$$\sum_{a\in[k_x]} \sigma_{a|x} = \bar{\sigma} \qquad \forall x \in [g]$$

  for some average state $\bar{\sigma}$
- Local hidden state (LHS) model:

$$\forall a \in [k_x],\ x \in [g], \qquad \sigma_{a|x} = \sum_{\lambda\in\Lambda} q_\lambda p(a|x,\lambda)\sigma_\lambda$$

Steering inequalities

- Steering inequality: tuples of self-adjoint matrices $\mathbf{F} := (F_{a|x})_{a,x}$
- LHS value:

$$V_{\mathcal{L}}(\mathbf{F}) := \sup_{\boldsymbol{\sigma} \in \mathcal{L}(g,\mathbf{k},d)} \sum_{a,x} \mathrm{Tr}(\sigma_{a|x} F_{a|x})$$

- Quantum value:

$$V_{\mathcal{Q}}(\mathbf{F}) := \sup_{\boldsymbol{\sigma} \in \mathcal{Q}(g,\mathbf{k},d)} \sum_{a,x} \mathrm{Tr}(\sigma_{a|x} F_{a|x})$$

- Restrict mostly to $k_x = 2$. In this case, $\mathbf{F}$ is unbiased if $F_{+|x} = -F_{-|x}$ for all $x \in [g]$

Steering constants

- Set of steering constants is defined as:

  $$\Sigma(g,d) := \{\mathbf{s} \in [0,1]^g \;:\; \forall (F_{\pm|1}, \dots, F_{\pm|g}) \in (\mathcal{M}_d^{\mathrm{sa}})^{2g},\; V_{\mathcal{L}}(\mathbf{F}) \leq 1 \implies V_{\mathcal{Q}}(\mathbf{F}^{(\mathbf{s})}) \leq 1\}$$

- $\mathbf{F}^{(\mathbf{s})} := \mathbf{s}.\mathbf{F} + (\mathbf{1} - \mathbf{s}).\mathbf{F}^{(\mathbf{0})}$ convex mixture with certain trivial steering inequality
- Quantifies how much steerability is available for fixed g, d
- Single number: largest quantum violation

  $$\gamma_{g,d} = \sup_{\mathbf{F}} \frac{V_{\mathcal{Q}}(\mathbf{F})}{V_{\mathcal{L}}(\mathbf{F})}$$

- $\Sigma_0(g,d)$, $\gamma^0_{g,d}$ after restricting to unbiased steering inequalities

Free spectrahedra

Free spectrahedra

- Free spectrahedra studied in optimization theory and algebraic convexity
- Fix tuple $\mathbf{A} \in (\mathcal{M}_d^{\mathrm{sa}})^{g+1}$. Free spectrahedron at level n

$$\hat{\mathcal{D}}_{\mathbf{A}}(n) := \left\{ (X_1, \ldots, X_g) \in (\mathcal{M}_n^{\mathrm{sa}})^g \ : \ \sum_{i \in [g]} A_i \otimes X_i \leq A_0 \otimes I_n \right\}.$$

- Free spectrahedron: $\hat{\mathcal{D}}_{\mathbf{A}} := \bigsqcup_{n \in \mathbb{N}} \hat{\mathcal{D}}_{\mathbf{A}}(n)$
- If $A_0 = I_d$, free spectrahedron is monic. Write $\mathcal{D}_{\mathbf{A}}$ instead

Matrix cube and matrix diamond

- *Matrix cube* $\mathcal{D}_{\square,g}$ [BTN02]:

$$\mathcal{D}_{\square,g}(n) := \{(X_1,\ldots,X_g) \in (\mathcal{M}_n^{\mathrm{sa}})^g \ : \ \|X_i\|_\infty \leq 1 \ \forall i \in [g]\}$$

- Can be written $\mathcal{D}_{\mathbf{A}}$ with $A_i = e_i \oplus -e_i \in \mathcal{M}_{2g}^{\mathrm{sa}}$ for all $i \in [g]$
- *Matrix diamond* $\mathcal{D}_{\diamond,g}$ [DDOSS17]. Its levels are defined as

$$\mathcal{D}_{\diamond,g}(n) := \{(X_1,\ldots,X_g\} \in (\mathcal{M}_n^{\mathrm{sa}})^g \ : \ \sum_{i\in[g]} \epsilon_i X_i \leq I_n \ \forall \boldsymbol{\epsilon} \in \{\pm 1\}^g\}$$

- Connected to compatibility of measurements in [BN18]

Inclusion constants

- Inclusion of free spectrahedra:

$$\hat{\mathcal{D}}_{\mathbf{A}} \subseteq \hat{\mathcal{D}}_{\mathbf{B}} \qquad \Longleftrightarrow \qquad \hat{\mathcal{D}}_{\mathbf{A}}(n) \subseteq \hat{\mathcal{D}}_{\mathbf{B}}(n) \quad \forall n \in \mathbb{N}$$

- Inclusion constant set:

$$\Delta_{\mathbf{A}}(g,d) := \{\mathbf{s} \in [0,1]^g \ : \ \mathcal{D}_{\mathbf{A}}(1) \subseteq \mathcal{D}_{\mathbf{B}}(1) \implies \mathbf{s}.\mathcal{D}_{\mathbf{A}} \subseteq \mathcal{D}_{\mathbf{B}} \text{ for all } \mathbf{B} \in (\mathcal{M}_d^{\mathrm{sa}})^g\}$$

- Write $\mathbf{s}.\mathcal{D} := \{(s_1X_1, \ldots, s_gX_g) \ : \ (X_1, \ldots, X_g) \in \mathcal{D}\}$
- Non-monic version:

$$\hat{\Delta}_{\mathbf{A}}(g,d) := \{\mathbf{s} \in [0,1]^g \ : \ \mathcal{D}_{\mathbf{A}}(1) \subseteq \hat{\mathcal{D}}_{\mathbf{B}}(1) \implies \mathbf{s}.\mathcal{D}_{\mathbf{A}} \subseteq \hat{\mathcal{D}}_{\mathbf{B}} \text{ for all } \mathbf{B} \in (\mathcal{M}_d^{\mathrm{sa}})^{g+1}\}$$

- Short-hand: $\Delta_{\square}(g,d)$ and $\hat{\Delta}_{\square}(g,d)$ for matrix cube

Maximal violation of steering inequalities

Connecting steering and the inclusion of the matrix diamond

- To any steering inequality $(F_{a|x})_{a,x}$, can associate free spectrahedron $\hat{\mathcal{D}}_{\tilde{\mathbf{F}}}(n)$

Theorem

For an arbitrary steering inequality $\mathbf{F}$, we have

$$\begin{aligned} &(1) \quad \mathcal{D}_{\square,g}(1) \subseteq \hat{\mathcal{D}}_{\tilde{\mathbf{F}}}(1) \iff V_{\mathcal{L}}(\mathbf{F}) \leq 1. \\ &(2) \quad \mathcal{D}_{\square,g} \subseteq \hat{\mathcal{D}}_{\tilde{\mathbf{F}}} \iff V_{\mathcal{Q}}(\mathbf{F}) \leq 1. \end{aligned}$$

Inclusion constants give maximal violation

Theorem

For all g, $d \in \mathbb{N}$, $\Sigma_0(g, d) = \Sigma(g, d) = \Delta_\square(g, d) = \hat{\Delta}_\square(g, d)$.

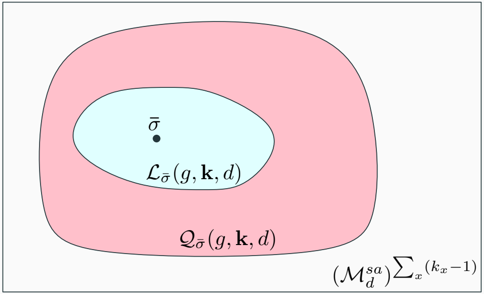
Proposition

For any $s \in [0, 1]$, $s(1, \dots, 1) \in \Sigma_0(g, d)$ if and only if for all unbiased (g, d)-steering inequalities $\mathbf{F}$ it holds that

$$V(\mathbf{F}) := \frac{V_{\mathcal{Q}}(\mathbf{F})}{V_{\mathcal{L}}(\mathbf{F})} \leq \frac{1}{s},$$

where we define $V(\mathbf{F}) = 1$ if $V_{\mathcal{L}}(\mathbf{F}) = V_{\mathcal{Q}}(\mathbf{F}) = 0$. In particular, the largest such s is equal to the largest unbiased quantum violation $\gamma^0_{g,d}$.

Inclusion constants as set inclusions



Proposition

We have, for all invertible density matrices $\bar{\sigma} \in \mathcal{M}_d^+$,

$$\{\mathbf{s} \in [0,1]^g \ : \ \mathbf{s}.\mathcal{Q}_{\bar{\sigma}}(g, 2^{\times g}, d) + (\mathbf{1} - \mathbf{s}).\boldsymbol{\bar{\sigma}} \subseteq \mathcal{L}_{\bar{\sigma}}(g, 2^{\times g}, d)\} = \Delta_{\square}(g, d).$$

g-independent bounds

Theorem

The largest s such that $s(1,\dots,1) \in \Delta_\square(g,d)$ for all $g \in \mathbb{N}$ is

$$\tau(d) = 4^{-n}\binom{2n}{n}, \qquad \text{with } n := \lfloor d/2 \rfloor.$$

Asymptotically, this behaves as $\sqrt{2/(\pi d)}$.

- It is possible to construct almost optimal steering inequalities, based on ϵ-nets of $\mathcal{U}(d)$.
- Based on the ideas in [HKMS19]

d-independent bounds

Proposition

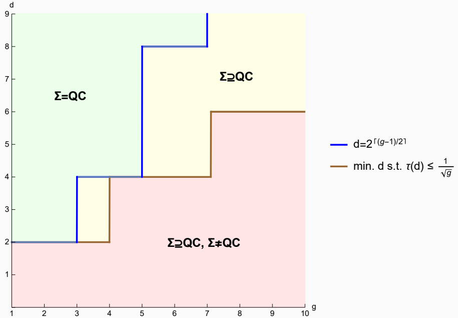
Let $d \geq 2^{\lceil (g-1)/2 \rceil}$, $g \geq 2$. Then,
$\Sigma(g,d) = \Delta_\square(g,d) = \left\{\mathbf{s} \in [0,1]^g \ : \ \sum_{i=1}^g s_i^2 \leq 1\right\}$. Generally, for any $d \in \mathbb{N}$,
$\Sigma(g,d) = \Delta_\square(g,d) \supseteq \left\{\mathbf{s} \in [0,1]^g \ : \ \sum_{i=1}^g s_i^2 \leq 1\right\}$.

- This shows that the unbiased steering inequalities considered in [MRY+15] are optimal
- For $d = 2$ Pauli matrices, in higher dimensions anticommuting self-adjoint unitaries are constructed by taking tensor powers

Implications

Phase diagram

$$\Sigma(g,d) := \{\mathbf{s} \in [0,1]^g \,:\, \forall (F_{\pm|1}, \ldots, F_{\pm|g}) \in (\mathcal{M}_d^{\mathrm{sa}})^{2g},\ V_{\mathcal{L}}(\mathbf{F}) \le 1 \implies V_{\mathcal{Q}}(\mathbf{F}^{(\mathbf{s})}) \le 1\}$$



$$\mathrm{QC}_g := \left\{ \mathbf{s} \in [0,1]^g \;:\; \sum_{i\in[g]} s_i^2 \le 1 \right\}$$

Conclusions

- Connection between quantum steering and the inclusion of the matrix cube
- Allows to place upper bound on the violation of dichotomic steering inequalities for fixed dimension d and fixed number of measurements on Alice's side g
- Shows previously found violations are optimal
- In particular

$$\frac{V_{\mathcal{Q}}(\mathbf{F})}{V_{\mathcal{L}}(\mathbf{F})} \leq \frac{4^n}{\binom{2n}{n}}$$

- Can be used as dimension witness

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