Incompatibility in general probabilistic theories, generalized spectrahedra, and tensor norms

Andreas Bluhm (University of Copenhagen)
— joint work with Anna Jenčová and Ion Nechita arXiv:2011.06497

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Talk outline

- Main topic: Incompatibility in general probabilistic theories

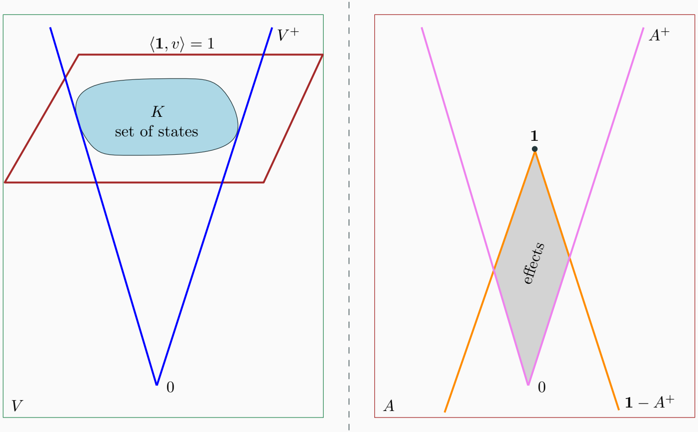
Question:

For a set of measurements, when is there a joint measurement implementing them all?

- Three different viewpoints: Positive maps, generalized spectrahedra, tensor norms

Compatibility in GPTs

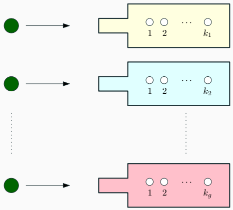
General Probabilistic Theories



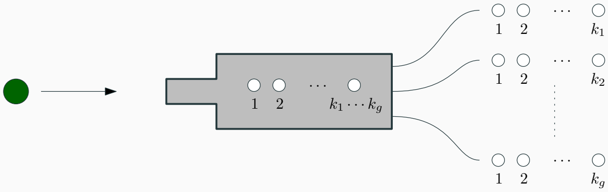
- A GPT is a triple $(V, V^+, \mathbb{1})$, where V is a vector space, $V^+ \subseteq V$ is a cone, and $\mathbb{1}$ is a linear form on V; $A = V^*$, $A^+ = (V^+)^*$, and $\mathbb{1} \in A^+$
- The set of states $K := V^+ \cap \mathbb{1}^{-1}(\{1\})$

Measurements and compatibility

- A GPT measurement with k outcomes is an affine map $K \to \Delta_k$
- A g-tuple of GPT measurements with $\mathrm{k} = (k_1, \dots, k_g)$ outcomes are encoded in an affine map $K \to \Delta_{k_1} \times \cdots \times \Delta_{k_g}$
- Measurements $f = (f^{(1)}, \dots, f^{(g)})$ are compatible if there exists a joint measurement h having $k_1 \cdots k_g$ outcomes such that



$$f_i^{(x)} = \sum_{j \in [k_1] \times \cdots \times [k_g] \,:\, j(x) = i} h_j \qquad \forall x \in [g],\ \forall i \in [k_x]$$



Tensor norms & compatibility

Injective and projective tensor norms

Definition

Consider Banach spaces X, Y. For a tensor $z \in X \otimes Y$, we define its projective tensor norm

$$\|z\|_\pi := \inf \left\{ \sum_{k=1}^{r} \|x_k\|_X \|y_k\|_Y \ :\ z = \sum_{k=1}^{r} x_k \otimes y_k,\ x_k \in X,\ y_k \in Y \right\}$$

and its injective tensor norm

$$\|z\|_\epsilon := \sup \{ |\langle \alpha \otimes \beta, z \rangle| \ :\ \alpha \in X^*,\ \beta \in Y^*,\ \|\alpha\|_{X^*} \leq 1,\ \|\beta\|_{Y^*} \leq 1 \}$$

- The projective and injective norms are examples of tensor norms:

$$\|x \otimes y\|_\rho = \|x\|_X \|y\|_Y \qquad \|\alpha \otimes \beta\|_{\rho^*} = \|\alpha\|_{X^*} \|\beta\|_{Y^*}$$

- For any tensor norm $\| \cdot \|_\rho$ on $X \otimes Y$, we have

$$\forall\ z \in X \otimes Y, \qquad \|z\|_\epsilon \leq \|z\|_\rho \leq \|z\|_\pi$$

Compatibility in terms of norms

- In our work, we restrict to dichotomic measurements
- Centrally symmetric GPTs: K is the unit ball of a norm $\|\cdot\|_{\bar{V}}$

$$V = \mathbb{R}v_0 \oplus \bar{V} \quad \text{and} \quad A = \mathbb{R}\mathbb{1} \oplus \bar{A}$$

Theorem

We identify a tensor norm $\|\cdot\|_\rho$ and tensors $\varphi^{(f)} \in \ell_\infty^g \otimes A$ such that

$$\|\varphi^{(f)}\|_\epsilon \leq 1 \iff \text{f effects}$$
$$\|\varphi^{(f)}\|_\rho \leq 1 \iff \text{f compatible.}$$

If the GPT is centrally symmetric and if f is unbiased, we can restrict to $\bar{\varphi}^{(f)} \in \ell_\infty^g \otimes \bar{A}$ and

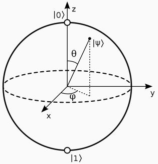
$$\|\bar{\varphi}^{(f)}\|_\rho = \|\bar{\varphi}^{(f)}\|_\pi.$$

Example: Bloch ball

- Qubit systems can be described as centrally symmetric GPTs with $\bar{V} = \bar{A} = \ell_2^3$.
- f is an effect if

$$f = \frac{1}{2}(\alpha I + \bar{a} \cdot \bar{\sigma})$$

  and $\|\bar{a}\|_2 \leq \alpha \leq 2 - \|\bar{a}\|_2$. The effect is unbiased if $\alpha = 1$.



- For two unbiased effects $f_1 = (1, \bar{a})$, $f_2 = (1, \bar{b})$:

$$\|\bar{\varphi}^{(f)}\|_\rho = \frac{1}{2}\|e_1 \otimes f_1 + e_2 \otimes f_2\|_{\ell_\infty^2 \otimes_\pi \ell_2^3} = \frac{1}{2}(\|\bar{a} + \bar{b}\|_2 + \|\bar{a} - \bar{b}\|_2)$$

- This criterion goes back to [Busch '86]
- $\|\cdot\|_\rho$-criterion generalizes this to arbitrary GPTs

Image source: https://commons.wikimedia.org/wiki/File:Bloch_sphere.svg

The compatibility region of a GPT

Compatibility regions and tensor norms

- Noisy version of GPT measurements (white noise)

$$(s.f)_i^{(x)} = s_x f_i^{(x)} + (1 - s_x)\frac{\mathbb{1}}{2}$$

- The set of noise parameters s rendering all measurements compatible is called the compatibility region

$$\Gamma(g; V, V^+) := \{s \in [0,1]^g \ : \ f \text{ measurements} \implies s.f \text{ compatible}\}$$

- Symmetric version: the compatibility degree

$$\gamma(g; V, V^+) := \max\{s \ : \ (s, s, \ldots, s) \in \Gamma(g; V, V^+)\}$$

Theorem

For dichotomic measurements in arbitrary GPTs, we have

$$\Gamma(g; V, V^+) = \{s \in [0,1]^g \ : \ \|s.\varphi\|_\rho \leq 1, \ \forall \varphi \in \ell_\infty^g \otimes A, \ \|\varphi\|_{\ell_\infty^g \otimes_\epsilon A} \leq 1\}.$$

The compatibility degree of a centrally symmetric GPT

Theorem

For dichotomic measurements in centrally symmetric GPTs, we have

$$\gamma(g; V, V^+) = 1/\rho(\ell_\infty^g, \bar{A})$$

where the quantity ρ was introduced in [Aubrun et al '20]

$$\rho(X, Y) = \max_{z \in X \otimes Y} \frac{\|z\|_{X \otimes_\pi Y}}{\|z\|_{X \otimes_\epsilon Y}}.$$

The maximally incompatible effects are unbiased.

Proposition

In the same setting as before

$$\lim_{g \to \infty} \gamma(g; V, V^+) = 1/\pi_1(\bar{V})$$

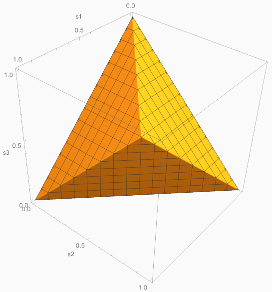
where $\pi_1(\bar{V})$ is the 1-summing norm of the Banach space $\bar{V}$.

Applications

- For the hypercubic GPT $\bar{V} = \ell_\infty^n$, we have

$$\Gamma(g; \ell_\infty^n) = \{s \in [0,1]^g \ : \forall I \subseteq [g] \text{ s.t. } |I| \leq n, \sum_{i \in I} s_i \leq 1\}$$

- We have $\gamma(g; \ell_\infty^n) = 1/\min(g, n)$



- Qubits: It was known that, for $g = 2, 3$, $\gamma(g; \mathrm{QM}_2) = 1/\sqrt{g}$

Proposition

For all $g \geq 4$,

$$0.5 \leq \gamma(g; \mathrm{QM}_2) \leq 1/\sqrt{3} \approx 0.577$$

Summary

- Compatibility of dichotomic measurements in general probabilistic theories is connected to tensor norms
- The criterion generalizes a well-known criterion for qubits
- For centrally symmetric GPTs, the compatibility degree is the ratio of extremal norms
- Allows to compute examples, e.g. new bounds on the compatibility degree of at least 4 qubit measurements