Compatibility of quantum measurements and inclusion constants for free spectrahedra

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Introduction

Compatibility of quantum measurements:

- Measurement = POVM
- Compatible if marginals of common measurement
- Only incompatible measurements can violate Bell inequalities
- Noise robustness quantifies incompatibility

Inclusion of free spectrahedra:

- Convex optimization
- Free spectrahedron = relaxation of linear matrix inequalities (dual SDPs)
- Inclusion constants quantify error

Aim of this talk: Connecting the two problems

Measurements

Quantum system described by a quantum state $\rho \in \mathcal{S}(\mathbb{C}^d)$,

$$\mathcal{S}(\mathbb{C}^d) := \{\rho \in \mathcal{M}_d : \rho \geq 0, \mathrm{Tr}(\rho) = 1\}.$$

Measurement:

- Measurement outcomes $\{a_i\}_{i=1}^m$, probabilities $\{p_i\}_{i=1}^m$
- Associate quantum state with probability:

$$p_i = \mathrm{Tr}(E_i \rho) \qquad \forall i \in \{1, \ldots, m\}$$

- E_i are effect operators:

$$\mathcal{E}(\mathbb{C}^d) := \{E \in \mathcal{M}_d : 0 \leq E \leq I_d\}$$

- Special case: Orthogonal projection $E^2 = E$
- Normalization:

$$I_d = \sum_{i=1}^m E_i$$

From now on: Measurement = Set of effect operators (POVM)

Compatibility: Two binary measurements

Example

Consider two binary measurements: $\{E, I - E\}$, $\{F, I - F\}$.
Assume that there is a measurement $\{R_{i,j}\}_{i,j=0}^{1}$ such that

$$\begin{array}{ccccc} R_{0,0} & + & R_{0,1} & = & E \\ + & & + & & \\ R_{1,0} & + & R_{1,1} & = & I - E \\ \| & & \| & & \\ F & & I - F & & \end{array}$$

Then the measurements are jointly measurable or compatible.

- For concrete measurements, this can be checked using an SDP.
- There is an equivalent definition via classical post processing.

The compatibility region

- Measurements can be made compatible by adding a sufficient amount of noise.
- White noise:

$$E \mapsto sE + \frac{1-s}{2} I_d, \qquad s \in [0,1]$$

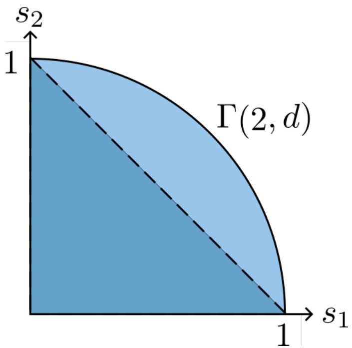
- Compatibility region:

$$\Gamma(g,d) := \Big\{ s \in [0,1]^g : \; s_i E_i + \frac{1-s_i}{2} I_d \text{ are compatible } \forall E_1, \ldots, E_g \in \mathcal{E}(\mathbb{C}^d) \Big\}$$

An easy example

Example

As $\Gamma(g,d)$ is convex, it holds $\left(\frac{1}{g},\ldots,\frac{1}{g}\right) \in \Gamma(g,d)\ \forall d \in \mathbb{N}$



$$\Gamma(g,d) := \Big\{ s \in [0,1]^g : s_i E_i + \frac{1-s_i}{2} I_d \text{ are comp.} \ \forall E_1,\ldots,E_g \in \mathcal{E}(\mathbb{C}^d) \Big\}.$$

Free spectrahedra

Let $A \in (M_d^{sa})^g$. The free spectrahedron at level n is defined as

$$\mathcal{D}_A(n) := \left\{ X \in \left(\mathcal{M}_n^{sa}\right)^g : \sum_{i=1}^{g} A_i \otimes X_i \leq I_{nd} \right\}.$$

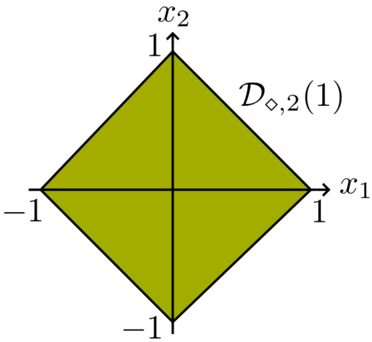
The free spectrahedron is the union of these levels

$$\mathcal{D}_A := \bigcup_{n \in \mathbb{N}} \mathcal{D}_A(n).$$

The matrix diamond

An important example is the matrix diamond:

$$\mathcal{D}_{\diamond,g}(n) = \left\{ X \in \left(\mathcal{M}_n^{sa}\right)^g : \sum_{i=1}^{g} \epsilon_i X_i \leq I_n \; \forall \epsilon \in \{\, -1, +1 \,\}^g \right\}.$$



Example

For $g = 2$:

$$A_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

n-positivity

- Let $\mathcal{L} \subseteq \mathcal{M}_D$ be a linear subspace such that $A \in \mathcal{L} \iff A^* \in \mathcal{L}$ and $I_D \in \mathcal{L}$.
- Let $\Phi : \mathcal{L} \to \mathcal{M}_d$ be a linear map.

Definition

We call Φ n-positive if $\Phi \otimes \mathrm{id}_n : \mathcal{L} \otimes \mathcal{M}_n \to \mathcal{M}_d \otimes \mathcal{M}_n$ is positive. The map Φ is completely positive if it is n-positive for all $n \in \mathbb{N}$.

- Complete positivity of Φ can be checked using an SDP[1].

[1] T. Heinosaari, M. A. Jivulescu, D. Reeb and M. M. Wolf. Extending quantum operations. *Journal of Mathematical Physics, 53(10):102208, 2012.*

Inclusion of free spectrahedra

- $\mathcal{D}_A \subseteq \mathcal{D}_B$ means $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$ for all $n \in \mathbb{N}$

Lemma[2]

Let $A \in (\mathcal{M}_D^{sa})^g$, $B \in (\mathcal{M}_d^{sa})^g$. Furthermore, let $\mathcal{D}_A(1)$ be bounded. The unital linear map $\Phi : \mathrm{span}\{I, A_1, \ldots, A_g\} \to \mathcal{M}_d^{sa}$,

$$\Phi : A_i \mapsto B_i \qquad \forall i \in [g]$$

is n-positive if and only if $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$.

- $\mathcal{D}_A(1) \subseteq \mathcal{D}_B(1) \implies s \cdot \mathcal{D}_A \subseteq \mathcal{D}_B$ for $s \in [0,1]^g$.
- Inclusion set: $\Delta(g,d) := \left\{ s \in [0,1]^g : \forall B \in \left(\mathcal{M}_d^{sa}\right)^g \right.$
  $\left. \mathcal{D}_{\diamond,g}(1) \subseteq \mathcal{D}_B(1) \Rightarrow s \cdot \mathcal{D}_{\diamond,g} \subseteq \mathcal{D}_B \right\}$

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[2] J. W. Helton et al. Dilations, linear matrix inequalities, the matrix cube problem and beta distributions. *Memoirs of the AMS*, 275(1232), 2019.

Connecting the two problems

Theorem

Let $E \in \left(\mathcal{M}_d^{sa}\right)^g$ and let $2E - I := (2E_1 - I_d, \dots, 2E_g - I_d)$. We have

1. $\mathcal{D}_{\diamond,g}(1) \subseteq \mathcal{D}_{2E-I}(1)$ *if and only if* $E_1, \dots, E_g$ *are effect operators.*
2. $\mathcal{D}_{\diamond,g} \subseteq \mathcal{D}_{2E-I}$ *if and only if* $E_1, \dots, E_g$ *are jointly measurable effect operators.*
3. $\mathcal{D}_{\diamond,g}(k) \subseteq \mathcal{D}_{2E-I}(k)$ *for* $k \in [d]$ *if and only if for any isometry* $V : \mathbb{C}^k \hookrightarrow \mathbb{C}^d$, *the induced compressions* $V^* E_1 V, \dots, V^* E_g V$ *are jointly measurable effect operators.*

Theorem

It holds that $\Gamma(g, d) = \Delta(g, d)$.

Proof ideas:

$\mathcal{D}_{\diamond,g}(1) \subseteq \mathcal{D}_{2E-I}(1)$ if and only if $E_1, \ldots, E_g$ are effect operators.

- Consider the extreme points $\pm e_i$ of the matrix diamond.

$\mathcal{D}_{\diamond,g} \subseteq \mathcal{D}_{2E-I}$ if and only if $E_1, \ldots, E_g$ are jointly measurable effect operators.

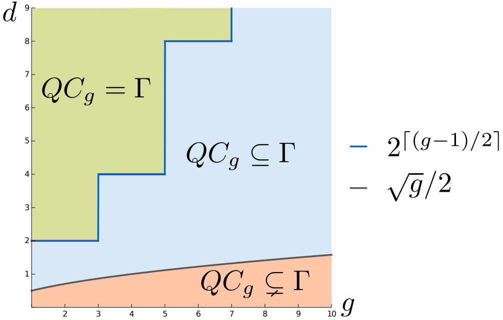
- Inclusion holds if and only if the unital map

$$\Phi : I_2^{\otimes(i-1)} \otimes \mathrm{diag}[-1, 1] \otimes I_2^{\otimes(g-i)} \mapsto 2E_i - I_d$$

  is completely positive
- Arveson's extension theorem: Φ has a positive extension $\tilde{\Phi}$ to $\mathbb{C}^{2^g}$
- Basis g_η of $\mathbb{C}^{2^g}$: $G_\eta := \tilde{\Phi}(g_\eta)$ is a joint POVM for $E_1, \ldots, E_g$ if and only if $\tilde{\Phi}$ positive

What we know about $\Gamma(g, d)$

$$\mathrm{QC}_g := \{s \in [0,1]^g : s_1^2 + \ldots + s_g^2 \leq 1\}$$



- Green: The upper and lower bound from Passer[3] coincide.
- Orange: Helton[4] shows $1/(2d)(1, \ldots, 1) \in \Gamma(g, d)$.

[2] B. Passer et al. Minimal and maximal matrix convex sets. *J. Funct. Anal.* , 274:3197–3253, 2018.

[4] J. W. Helton et al. Dilations, linear matrix inequalities, the matrix cube problem and beta distributions. *Memoirs of the AMS*, 275(1232), 2019.

Conclusion

- Compatibility of binary POVMs corresponds to inclusion of the matrix diamond into a free spectrahedron defined by the effect operators
- Compatibility region = Inclusion set of the matrix diamond
- $\Gamma(g, d) = QC_g$ for dimension d exponential in the number of measurements g

References:

1. AB and Ion Nechita. Joint measurability of quantum effects and the matrix diamond. *Journal of Mathematical Physics*, 59(11):112202, 2018.
2. AB and Ion Nechita. Compatibility of quantum measurements and inclusion constants for the matrix jewel. *arXiv1809.04514*, 2018.

Points in the inclusion set

It holds that $\Gamma(g, d) = \Delta(g, d)$.

- Davidson et al.[5]: Point independent of d

$$\frac{1}{g}(1, \ldots, 1) \in \Delta(g, d)$$

- Helton et al.[6]: Point independent of g

$$\frac{1}{2d}(1, \ldots, 1) \in \Delta(g, d)$$

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[2] K. R. Davidson et al. Dilations, inclusions of matrix convex sets, and completely positive maps. *Int. Math. Res. Notices*, 2017(13):4069–4130, 2017.

[6] J. W. Helton et al. Dilations, linear matrix inequalities, the matrix cube problem and beta distributions. *arXiv:1412.1481*, 2014.

Upper and lower bounds for the matrix diamond [7]

Theorem

Let $g, d \in \mathbb{N}$. Then, it holds that $\mathrm{QC}_g \subseteq \Delta(g,d)$. In other words, for any g-tuple $E_1, \ldots, E_g$ of effect operators and any positive vector $s \in \mathbb{R}^g_+$ with $\|s\|_2 \leq 1$, the g-tuple of noisy effect operators $E_i' = s_i E_i + (1 - s_i) I_d / 2$ is jointly measurable.

Theorem

Let $g \geq 2$, $d \geq 2^{\lceil (g-1)/2 \rceil}$. Then, $\Delta(g,d) \subseteq \mathrm{QC}_g$.

$$QC_g := \{s \in [0,1]^g : s_1^2 + \ldots + s_g^2 \leq 1\}$$

[4] B. Passer et al. Minimal and maximal matrix convex sets. *J. Funct. Anal.* , 274:3197–3253, 2018.

Maximally incompatible measurements

We can construct effect operators which achieve the upper bound:

$$F_i^{(k+1)} = \sigma_X \otimes F_i^{(k)} \qquad \forall i \in [2k+1]$$

$$F_{2k+2}^{(k+1)} = \sigma_Y \otimes I_{2^k}, \qquad F_{2k+3}^{(k+1)} = \sigma_Z \otimes I_{2^k}.$$

Example

$k = 1$: $F_1^{(1)} = \sigma_X$, $F_2^{(1)} = \sigma_Y$, $F_3^{(1)} = \sigma_Z$
$k = 2$:

$$F_1^{(2)} = \sigma_X \otimes \sigma_X, \qquad F_2^{(2)} = \sigma_X \otimes \sigma_Y, \qquad F_3^{(2)} = \sigma_X \otimes \sigma_Z,$$

$$F_4^{(2)} = \sigma_Y \otimes I_2, \qquad F_5^{(2)} = \sigma_Z \otimes I_2$$

Outlook: More outcomes

The matrix diamond is the universal for binary measurements, which object do we consider for more outcomes?

- Line with endpoints ± 1 is a simplex S_1 in one dimension
- $\mathcal{D}_{\diamond,2}(1) = S_1 \oplus S_1$
- Measurements with k-outcomes: S_{k-1}
- Level 1: $S_{k_1-1} \oplus \ldots \oplus S_{k_g-1}$
- Matrix diamond is the maximal free spectrahedron sitting on the ℓ_1-ball
- Taking the maximal free spectrahedron for k-outcomes leads to the matrix jewel
- Connection carries over to the general setting
- Similar inclusion problems can be found for the compatibility of quantum channels and compatibility in GPTs (ongoing)