Free spectrahedra in quantum information theory

Andreas Bluhm
Univ. Grenoble Alpes, CNRS, Grenoble INP, LIG

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Talk outline

Measurement compatibility

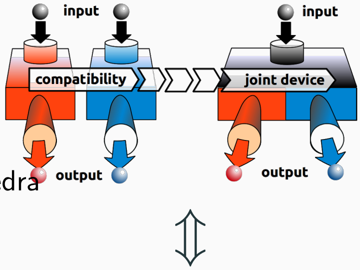
Free spectrahedra

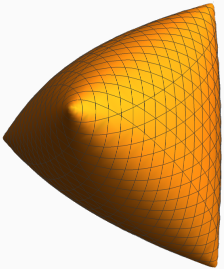
Linking measurement compatibility and free spectrahedra

Matrix convex sets

Linking measurement compatibility and matrix convex sets

Polytope compatibility





joint work with I. Nechita and S. Schmidt

Measurement compatibility

Measurements

Measurement of a quantum system in state ρ:

- Measurement outcomes $\{a_i\}_{i=1}^k$, probabilities $\{p_i\}_{i=1}^k$
- Effect operators:

$$\mathrm{Eff}_D := \{E \in \mathcal{M}_d(\mathbb{C}) : 0 \leq E \leq I_d\}$$

- Special case: Orthogonal projection $E^2 = E$
- Associate with probability:

$$p_i = \mathrm{tr}[E_i \rho] \qquad \forall i \in [k]$$

- Normalization:

$$I = \sum_{i=1}^{k} E_i$$

Measurements give the probabilities of the classical outcomes when a quantum state enters a measurement apparatus. Mathematically, they are modeled by POVMs.

Quantum measurements: Compatibility

Definition

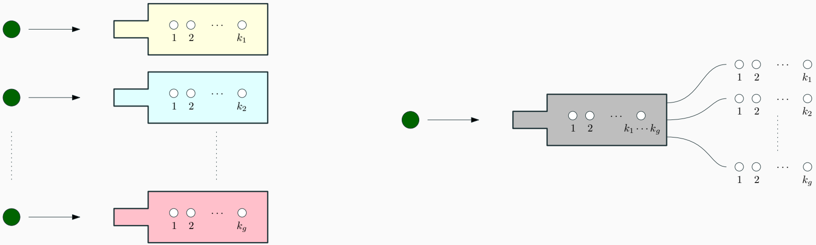
Two POVMs, $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_l)$, are called compatible if there exists a third POVM $C = (C_{ij})_{i\in[k], j\in[l]}$ such that

$$\forall i \in [k], \quad A_i = \sum_{j=1}^{l} C_{ij} \qquad \text{and} \qquad \forall j \in [l], \quad B_j = \sum_{i=1}^{k} C_{ij}.$$

The definition generalizes to g-tuples of POVMs $A^{(1)}, \ldots, A^{(g)}$, having respectively $k_1, \ldots k_g$ outcomes, where the joint POVM C has outcome set $[k_1] \times \cdots \times [k_g]$.

- Other way to say that: jointly measurable.
- Compatibility of measurements can be checked using a semidefinite program

What does it mean?



- Examples:
  1. Trivial POVMs $A = (p_i I_d)$ and $B = (q_j I_d)$ are compatible.
  2. Commuting POVMs $[A_i, B_j] = 0$ are compatible.
  3. If the POVM A is projective, then A and B are compatible if and only if they commute.
- Only incompatible measurements can show Bell inequality violations
- Incompatibility is hence a resource for quantum information processing

Compatibility: Alternative definition

There is an alternative (equivalent) definition via post-processing[1]:

Lemma

Let $E^{(j)} \in (\mathcal{M}_d^{sa}(\mathbb{C}))^{k_j}$, $j \in [g]$, be a collection of POVMs. These POVMs are jointly measurable if and only if there is some $m \in \mathbb{N}$ and a POVM $M \in (\mathcal{M}_d^{sa}(\mathbb{C}))^m$ such that

$$E_i^{(j)} = \sum_{x=1}^{m} p_j(i|x) M_x$$

for all $i \in [k_j]$, $j \in [g]$ and some conditional probabilities $p_j(i|x)$.

Compatible measurements can be simulated by a single joint measurement, by classically post-processing its outputs .

[1] T. Heinosaari et al. *An invitation to quantum incompatibility*. J. Phys. A, 49(12), 2016.

Noisy POVMs

- POVMs can be made compatible by adding noise, i.e. mixing in trivial POVMs.
- Example: dichotomic POVMs and white noise, $s \in [0,1]$:

$$(E, I - E) \mapsto s(E, I - E) + (1 - s)(\frac{I}{2}, \frac{I}{2}) \qquad \text{or} \qquad E \mapsto sE + (1 - s)\frac{I}{2}.$$

- Taking $s = 1/2$ suffices to render any pair of dichotomic POVMs compatible $\rightsquigarrow$ define $C_{ij} := (E_i + F_j)/4$.
- From now on, we focus on dichotomic (YES/NO) POVMs.

Definition

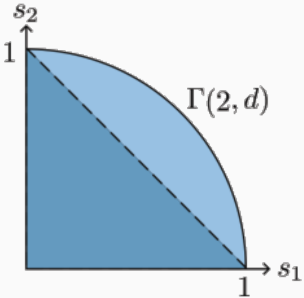
The compatibility region for g measurements on $\mathbb{C}^d$ is the set

$$\Gamma(g, d) := \{s \in [0,1]^g \ : \ \text{for all quantum effects } E_1, \ldots, E_g \in \mathcal{M}_d(\mathbb{C}), \\ \text{the noisy versions } s_i E_i + (1 - s_i) I_d/2 \text{ are compatible}\}$$

Compatibility region

$$\Gamma(g,d) := \{s \in [0,1]^g \ : \text{ for all quantum effects } E_1,\ldots,E_g \in \mathcal{M}_d(\mathbb{C}),$$
$$\text{the noisy versions } s_iE_i + (1-s_i)I_d/2 \text{ are compatible}\}$$

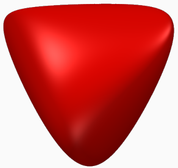
- The set $\Gamma(g,d)$ is convex.
- For all $i \in [g]$, $e_i \in \Gamma(g,d)$: every measurement is compatible with $g-1$ trivial measurements.
- For $d \geq 2$, $(1,1,\ldots,1) \notin \Gamma(g,d)$: there exist incompatible measurements.
- For all $d \geq 2$, $\Gamma(2,d)$ is a quarter-circle.



Generally speaking, the set $\Gamma(g,d)$ tells us how robust (to noise) is the incompatibility of g dichotomic measurements on $\mathbb{C}^d$.

Free spectrahedra

Free spectrahedra

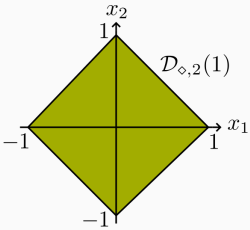


- A spectrahedron is given by PSD constraints: for $A = (A_1, \dots, A_g) \in (\mathcal{M}_d^{\mathrm{sa}}(\mathbb{C}))^g$
$$\mathcal{D}_A(1) := \left\{ x \in \mathbb{R}^g \; : \; \sum_{i=1}^{g} x_i A_i \leq I_d \right\}$$
  - $\mathcal{D}_{(\sigma_X, \sigma_Y, \sigma_Z)}(1) = \{(x, y, z) \in \mathbb{R}^3 \; : \; x\sigma_X + y\sigma_Y + z\sigma_Z \leq I_2\} = \text{Bloch ball}$
  - A free spectrahedron is the matricization of a spectrahedron
$$\mathcal{D}_A := \bigsqcup_{n=1}^{\infty} \mathcal{D}_A(n) \quad \text{with} \quad \mathcal{D}_A(n) := \left\{ X \in (\mathcal{M}_n^{sa}(\mathbb{C}))^g \; : \; \sum_{i=1}^{g} X_i \otimes A_i \leq I_{nd} \right\}$$

Example: the matrix diamond

The matrix diamond is the free spectrahedron defined by

$$\mathcal{D}_{\Diamond,g} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{\mathrm{sa}}(\mathbb{C}))^g \,:\, \sum_{i=1}^{g} \epsilon_i X_i \leq I_n, \quad \forall \epsilon \in \{\pm 1\}^g\}$$



- At level one, $\mathcal{D}_{\Diamond,g}(1)$ is the unit ball of the ℓ^1 norm on $\mathbb{R}^g$
- As a free spectrahedron, it is defined by $2^g \times 2^g$ diagonal matrices $\mathcal{D}_{\Diamond,g} = \mathcal{D}_{L_1,\ldots,L_g}$, with $L_i = I_2 \otimes \cdots \otimes I_2 \otimes \mathrm{diag}(1,-1) \otimes I_2 \otimes \cdots \otimes I_2$

Spectrahedral inclusion

- Consider two free spectrahedra defined by $(A_1, \dots, A_g)$ and $(B_1, \dots, B_g)$
- We write $\mathcal{D}_A \subseteq \mathcal{D}_B$ if, for all $n \geq 1$, $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$
- Clearly, $\mathcal{D}_A \subseteq \mathcal{D}_B \implies \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$. For the converse implication to hold, one may need to shrink $\mathcal{D}_A$...

Definition

For a free spectrahedron $\mathcal{D}_A$, we define its set of inclusion constants as

$$\Delta_A(g, d) := \{s \in [0,1]^g \; : \text{for all } g\text{-tuples } B_1, \dots, B_g \in \mathcal{M}_d(\mathbb{C})^{\mathrm{sa}}, \\ \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1) \implies s.\mathcal{D}_A \subseteq \mathcal{D}_B\}$$

- We shall be concerned with the inclusion set for the matrix diamond, which we denote by $\Delta(g, d)$

Spectrahedral inclusion continued

- The inclusion constants for the matrix cube play an important role in combinatorial optimization
- There, one is interested in whether $\mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$
- However, that is an NP-hard problem
- On the other hand, $\mathcal{D}_A \subseteq \mathcal{D}_B$ can be checked with a semidefinite program
- The latter is hence a matricial relaxation
- Inclusion constants quantify how good this relaxation is

Linking measurement compatibility and free spectrahedra

Compatibility in QM $\iff$ matrix diamond inclusion

To a g-tuple $E \in (\mathcal{M}_d^{\mathrm{sa}}(\mathbb{C}))^g$, we associate:

$$\mathcal{D}_{2E-I} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{\mathrm{sa}}(\mathbb{C}))^g \ : \ \sum_{i=1}^{g} X_i \otimes (2E_i - I_d) \leq I_{nd}\}$$

Theorem

Let $E \in (\mathcal{M}_d^{sa}(\mathbb{C}))^g$ be g-tuple of selfadjoint matrices. Then:

- *The matrices E are quantum effects* $\iff$ $\mathcal{D}_{\diamondsuit,g}(1) \subseteq \mathcal{D}_{2E-I}(1)$
- *The matrices E are compatible quantum effects* $\iff$ $\mathcal{D}_{\diamondsuit,g} \subseteq \mathcal{D}_{2E-I}$

At the intermediate levels $1 \leq n \leq d$, $\mathcal{D}_{\diamondsuit,g}(n) \subseteq \mathcal{D}_{2E-I}(n)$ iff for all isometries $V : \mathbb{C}^n \to \mathbb{C}^d$, the compressed effects $V^* E_i V$ are compatible.

Moreover, the compatibility region is equal to the set of inclusion constants of the matrix diamond: $\forall g, d$, $\Gamma(g,d) = \Delta(g,d)$.

Consequences

Many things are known about the matrix diamond:

- For all g, d, $\frac{1}{2d}(1, 1, \ldots, 1) \in \Delta(g, d)$ (Helton *et al.*, 2019)[2]
- For all g, d, $\mathrm{QC}_g := \{s \in [0, 1]^g \; : \; \sum_i s_i^2 \leq 1\} \subseteq \Delta(g, d)$ (Passer *et al.*, 2018)[3]

Theorem

For all g and $d \geq 2^{\lceil (g-1)/2 \rceil}$, $\Gamma(g, d) = \Delta(g, d) = \mathrm{QC}_g$

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[2] J. W. Helton, I. Klep, S. A. McCullough, M. Schweighofer: *Dilations, linear matrix inequalities, the matrix cube problem and beta distributions*. Mem. Amer. Math. Soc. 257 vol. 1232, 2019

[3] B. Passer, O. Shalit, B. Solel: *Minimal and maximal matrix convex sets*. J. Funct. Anal. 274(11), 2018

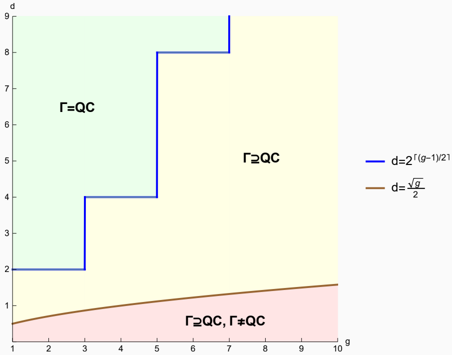
More consequences

Many things are known about (in-)compatibility:

- Some small g, d cases completely solved
- Approximate quantum cloning $\implies$ compatibility

$$\mathrm{Clone}(g,d) := \{s \in [0,1]^g : \exists \text{ quantum channel } \Phi : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})^{\otimes g} \text{ s.t.}$$
$$\forall i \in [g], \quad \Phi_i(X) = s_i X + (1 - s_i)\frac{\operatorname{Tr} X}{d} I_d\}$$

Phase diagram



- Connection to free spectrahedra also holds for arbitrary outcomes
- Instead of matrix diamond, consider its generalization, the matrix jewel
- Similar ideas can be used in general probabilistic theories
- We can get a better lower curve later

Inclusion of spectrahedra and (completely) positive maps

Theorem (Helton et al., 2013)

Let $A \in (\mathcal{M}_D^{sa}(\mathbb{C}))^g$, $B \in (\mathcal{M}_d^{sa}(\mathbb{C}))^g$ such that $\mathcal{D}_A(1)$ is bounded. Then, $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$ iff the unital linear map

$$\Phi : \mathrm{span}\{I, A_1, \ldots, A_g\} \to \mathcal{M}_d^{sa}(\mathbb{C}), \qquad A_i \mapsto B_i$$

is n-positive.

Sketch of the proof of the main theorem:

- Level 1: the extremal points of $\mathcal{D}_{\diamondsuit,g}(1)$ are $\pm e_i$
- The inclusion $\mathcal{D}_{\diamondsuit,g} \subseteq \mathcal{D}_{2E-I}$ holds iff the unital map $\Phi : I_2 \otimes \cdots \otimes I_2 \otimes \mathrm{diag}(1,-1) \otimes I_2 \otimes \cdots \otimes I_2 \mapsto 2E_i - I_d$ is CP
- Arveson's extension theorem: Φ has a (completely) positive extension $\tilde{\Phi}$ to $\mathbb{R}^{2^g}$
- $C_f := \tilde{\Phi}(f)$ is a joint POVM for the E_i's, where $\{f\}$ is a basis of $\mathbb{R}^{2^g}$

Maximally incompatible quantum effects

Lemma (Newman 1932, Hrubeš 2016)

For $d = 2^k$, there exist $2k+1$ anti-commuting, self-adjoint, unitary matrices $F_1, \ldots, F_{2k+1} \in \mathcal{U}_d$. Moreover, 2^k is the smallest dimension where such a $(2k+1)$-tuple exists.

- For $k = 0$, take $F_1^{(0)} := [1]$
- For $k \geq 1$, define $F_i^{(k+1)} = \sigma_X \otimes F_i^{(k)}$ $\forall i \in [2k+1]$ and $F_{2k+2}^{(k+1)} = \sigma_Y \otimes I_{2^k}$, $F_{2k+3}^{(k+1)} = \sigma_Z \otimes I_{2^k}$
- These matrices satisfy, for all $x \in \mathbb{R}_+^g$, $\left\|\sum_{i=1}^g x_i F_i\right\|_\infty = \|x\|_2$, and $\left\|\sum_{i=1}^g x_i \bar{F}_i \otimes F_i\right\|_\infty = \|x\|_1$
- For d large enough, the maximally incompatible g-tuple of quantum effects in $\mathcal{M}_d(\mathbb{C})$ is given by $E_i = (F_i + I_d)/2$

Matrix convex sets

Matrix convex sets

We consider free sets:

$$\mathcal{F} = \bigsqcup_{i \in \mathbb{N}} \mathcal{F}_i,$$

where $\mathcal{F}_i \subseteq (\mathcal{M}_i^{\mathrm{sa}}(\mathbb{C}))^g$.

The free set $\mathcal{F}$ is matrix convex if it is closed under direct sums and unital completely positive maps:

- $(A_1, \ldots, A_g) \in \mathcal{F}_i$, $(B_1, \ldots, B_g) \in \mathcal{F}_j \implies (A_1 \oplus B_1, \ldots, A_g \oplus B_g) \in \mathcal{F}_{i+j}$.
- $(A_1, \ldots, A_g) \in \mathcal{F}_i$, $\Phi : \mathcal{M}_i(\mathbb{C}) \to \mathcal{M}_j(\mathbb{C})$ UCP $\implies (\Phi(A_1), \ldots, \Phi(A_g)) \in \mathcal{F}_j$

Free spectrahedra are special cases of matrix convex sets

Minimal and maximal matrix convex sets

- Unless $\mathcal{F}_1$ is a simplex, there are arbitrarily many different matrix convex sets with the same $\mathcal{F}_1$. However, there is a largest and a smallest such set:
- For a closed convex set $\mathcal{C}$,

$$\mathcal{W}_n^{\max}(\mathcal{C}) := \left\{ X \in (\mathcal{M}_n^{\mathrm{sa}}(\mathbb{C}))^g : \sum_{i=1}^{g} c_i X_i \leq \alpha I \ \forall (\alpha, c) \text{ supp. hyperplanes for } \mathcal{C} \right\}$$

- For a closed convex set $\mathcal{C}$,

$$\mathcal{W}_n^{\min}(\mathcal{C}) := \left\{ X = \sum_j z_j \otimes Q_j \in (\mathcal{M}_n^{\mathrm{sa}}(\mathbb{C}))^g : z_j \in \mathcal{C},\ Q_j \geq 0 \ \forall j, \sum_j Q_j = I_n \right\}$$

- Observe $\mathcal{W}_1^{\max}(\mathcal{C}) = \mathcal{C} = \mathcal{W}_1^{\min}(\mathcal{C})$. $\mathcal{W}^{\max}(\mathcal{C})$ quantizes hyperplanes, $\mathcal{W}^{\min}(\mathcal{C})$ quantizes extreme points.
- The matrix diamond is a maximal matrix convex set, $\mathcal{D}_{\diamond,g} = \mathcal{W}^{\max}(\mathcal{B}(\ell_1^g))$

Inclusion sets

Definition

Let $d, g \in \mathbb{N}$ and $\mathcal{C} \subset \mathbb{R}^g$ closed convex. The inclusion set is defined as

$$\Delta_{\mathcal{C}}(d) := \left\{ s \in [0,1]^g : s \cdot \mathcal{W}_d^{\max}(\mathcal{C}) \subseteq \mathcal{W}_d^{\min}(\mathcal{C}) \right\}.$$

If $\mathcal{C}$ is the ℓ_∞^g unit ball, we write $\Delta_\square(g,d)$.

Depending on the set $\mathcal{C}$, sometimes bounds on the inclusion set are known.

Linking measurement compatibility and matrix convex sets

Measurement compatibilty revisited

From now on, we concentrate on measurements with two outcomes and identify $E^{(i)} = \{E_i, I - E_i\}$ with E_i.

Theorem

Let

$$A = \sum_{j=1}^{g} e_j \otimes (2E_j - I).$$

Then,

1. $A \in \mathcal{W}_d^{\max}(\mathcal{B}(\ell_\infty^g))$ *if and only if* $\{E_j\}_{j\in[g]}$ *is a collection of POVMs.*
2. $A \in \mathcal{W}_d^{\min}(\mathcal{B}(\ell_\infty^g))$ *if and only if* $\{E_j\}_{j\in[g]}$ *is a collection of compatible POVMs.*

Proof sketch

- $\mathcal{W}_d^{\mathrm{max}}(\mathcal{B}(\ell_\infty^g))$ is given in terms of hyperplanes. Have to verify $-I \leq A_i = 2E_i - I \leq I$ $\implies 0 \leq E_i \leq I$.
- Reminder:

$$\mathcal{W}_n^{\mathrm{min}}(\mathcal{B}(\ell_\infty^g)) := \Big\{X = \sum_j z_j \otimes Q_j \in (\mathcal{M}_n^{\mathrm{sa}}(\mathbb{C}))^g : z_j \in \mathcal{C}\ \forall j,\ Q \text{ POVM}\Big\}.$$

- Going to extreme points:

$$2E_j - I = \sum_{\epsilon \in \{\pm 1\}^g} \epsilon(j) Q_\epsilon.$$

- Using $\sum_\epsilon Q_\epsilon = I$:

$$E_j = \sum_{\epsilon \in \{\pm 1\}^g} \delta_{\epsilon(j),1} Q_\epsilon.$$

- $\{Q_\epsilon\}_\epsilon$ is a joint POVM.

Inclusion sets and compatibility regions

Theorem

Let $g, d \in \mathbb{N}$. Let $s \in [0,1]^g$. Then, $\{s_i E_i + (1-s_i)I/2\}_{i\in[g]}$ is a collection of compatible POVMs for all POVMs $\{E_i\}_{i\in[g]}$, if and only if $s \in \Delta_\square(g,d)$. An equivalent way to phrase this is $\Gamma(g,d) = \Delta_\square(g,d)$.

- This follows from the computation

$$A_i' = 2(s_i E_i + (1-s_i)I/2) - I = s_i(2E_i - I) = s_i A_i.$$

- So adding noise means scaling the tensor A and hence $s \cdot \mathcal{W}_d^{\max}(\mathcal{B}(\ell_\infty^g))$ is the set of noisy measurements.
- Thus, $s \cdot A \in \mathcal{W}_d^{\min}(\mathcal{B}(\ell_\infty^g))$ means the noisy measurements are compatible.

g-independent bounds

Theorem

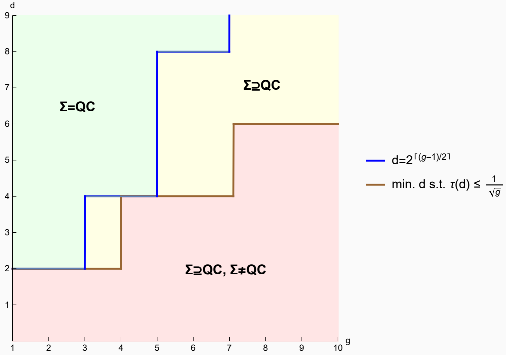
The largest s such that $s(1,\dots,1) \in \Delta_\square(g,d)$ for all $g \in \mathbb{N}$ is

$$\tau(d) = 4^{-n}\binom{2n}{n}, \qquad \text{with } n := \lfloor d/2 \rfloor.$$

Asymptotically, this behaves as $\sqrt{2/(\pi d)}$.

- It is possible to show that this is asymptotically optimal, based on ϵ-nets of $\mathcal{U}(d)$.
- Based on the ideas in Helton *et al.*, which in turn were inspired by Ben-Tal and Nemirovski

Improved phase diagram



This improves over the lower curve shown earlier.

Polytope compatibility

Polytope compatibility

Definition

Let $\mathcal{P}$ be a polytope in $\mathbb{R}^g$ such that $0 \in \operatorname{int} \mathcal{P}$. Let

$$A = (A_1, \ldots, A_g) \in \mathcal{M}_d^{\mathrm{sa}}(\mathbb{C})^g \cong \mathbb{R}^g \otimes \mathcal{M}_d^{sa}(\mathbb{C})$$

a g-tuple of Hermitian matrices. Then, A are *$\mathcal{P}$-operators* if and only if $A \in \mathcal{W}_d^{\max}(\mathcal{P})$. Moreover, A are *$\mathcal{P}$-compatible* if and only if $A \in \mathcal{W}_d^{\min}(\mathcal{P})$.

Motivation:

- A are $\mathcal{B}(\ell_\infty^g)$-operators if and only if $\frac{1}{2}(A_i + I)$ are dichotomic POVMs.
- A are $\mathcal{B}(\ell_\infty^g)$-compatible if and only if $\frac{1}{2}(A_i + I)$ are compatible dichotomic POVMs.

Magic squares

A magic square is a collection of positive operators A_{ij}, $i, j \in [N]$, such that

$$
\begin{array}{ccccccccc}
A_{11} & + & A_{12} & + & \dots & + & A_{1N} & = I \\
+ & & + & & & & + & \\
\vdots & & \vdots & & \ddots & & \vdots & \vdots \\
+ & & + & & & & + & \\
A_{N1} & + & A_{N2} & + & \dots & + & A_{NN} & = I \\
\| & & \| & & & & \| & \\
I & & I & & \dots & & I &
\end{array}
$$

The magic square is said to be semiclassical if

$$
A = \sum_{i,j\in[N]} |i\rangle\langle j| \otimes A_{ij} = \sum_{\pi\in\mathcal{S}_N} P_\pi \otimes Q_\pi,
$$

where P_π is the permutation matrix associated to π and $\{Q_\pi\}_\pi$ is a POVM.

Birkhoff polytope compatibility

Definition

For a given $N \geq 2$, the Birkhoff body $\mathcal{B}_N(1)$ is defined as the set of $(N-1) \times (N-1)$ truncations of $N \times N$ bistochastic matrices, shifted by J/N:

$$\mathcal{B}_N = \{A^{(N-1)} - J_{N-1}/N \, : \, A \in \mathcal{M}_N(\mathbb{R}) \text{ bistochastic}\} \subset \mathbb{R}^{(N-1)^2}.$$

Theorem

Consider a $(N-1)^2$-tuple of selfadjoint matrices $A \in \mathcal{M}_d^{\mathrm{sa}}(\mathbb{C})^{(N-1)^2}$ and the corresponding matrix $\tilde{A} \in \mathcal{M}_N(\mathcal{M}_d(\mathbb{C}))$. Then:

1. *The matrix $\tilde{A}$ is a magic square if and only if $A - I/N$ are $\mathcal{B}_N$-operators.*
2. *The matrix $\tilde{A}$ is a semiclassical magic square if and only if $A - I/N$ are $\mathcal{B}_N$-compatible.*

Relation to measurement incompatibility

Is being a semiclassical magic square the same as being compatible? No.

$\frac{1}{2}\lvert 0\rangle\langle 0\rvert$	$\frac{1}{2}\lvert 1\rangle\langle 1\rvert$	0	$\frac{1}{2}I_2$
$\frac{1}{2}\lvert 1\rangle\langle 1\rvert$	$\frac{1}{2}\lvert 0\rangle\langle 0\rvert$	$\frac{1}{2}I_2$	0
0	$\frac{1}{2}I_2$	$\frac{1}{2}\lvert +\rangle\langle +\rvert$	$\frac{1}{2}\lvert -\rangle\langle -\rvert$
$\frac{1}{2}I_2$	0	$\frac{1}{2}\lvert -\rangle\langle -\rvert$	$\frac{1}{2}\lvert +\rangle\langle +\rvert$

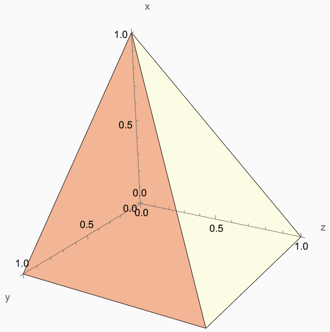
These measurements are compatible, but they do not form a semiclassical magic square.

Reason: $\mathcal{B}_N$-compatibility restricts the post-processing to $p_i(j|\lambda) = p_j(i|\lambda)$, i.e., enforces special structure in the joint POVM.

Measurement compatibility with shared effects

Can we generalize the magic square example?

$$\mathcal{P} = (-1/3, -1/3, -1/3) + \mathrm{conv}\{((1,0,0), (0,0,0), (0,0,1), (0,1,0), (0,1,1)\}.$$



Consider $(A, B, C) \in (\mathcal{M}_d(\mathbb{C})^{sa})^3$. Then, we have $(A, B, C) + 1/3(I, I, I) \in \mathcal{W}_d^{\max}(\mathcal{P})$ if and only if both $(A, B, I_d - A - B)$ and $(A, C, I_d - A - C)$ are POVMs.

Measurement compatibility with shared effects, continued

When does $(A, B, C) + 1/3(I, I, I) \in \mathcal{W}_d^{\min}(\mathcal{P})$ hold? Equivalent to the existence of a joint measurement such that

Q_1	0	0	$= A$
0	Q_5	Q_4	$= B$
0	Q_3	Q_2	$= I_d - A - B$
$= A$	$= C$	$= I_d - A - C$	

Not all joint measurements are of this form, check

$$\left(\frac{1}{2} I_2, \frac{1}{2} |0\rangle\langle 0| , \frac{1}{2} |1\rangle\langle 1|\right) \qquad \text{and} \qquad \left(\frac{1}{2} I_2, \frac{1}{2} |+\rangle\langle +| , \frac{1}{2} |-\rangle\langle -|\right).$$

As we will see now, polytope compatibility can always be seen as measurement compatibility with shared effects and restricted post-processing.

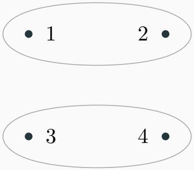
Probability hypergraphs

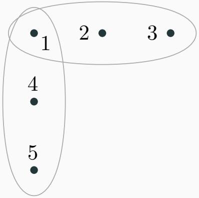
Definition

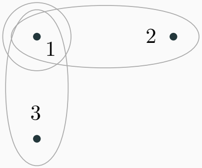
A hypergraph G (with no isolated vertices) is called a probability hypergraph if there exists a function $\pi : V \to (0,1]$ such that

$$\forall e \in E, \quad \sum_{v \in e} \pi(v) = 1. \tag{1}$$

We denote by $\Pi(G)$ the set of all functions $\pi : V \to [0,1]$ such that Eq. (1) holds.







Polytopes from probability hypergraphs

We associate to a probability hypergraph G a polytope in an essentially unique manner:

$$\Pi^0(G) := \{\pi : V \to \mathbb{R} : \forall e \in E, \sum_{v \in e} \pi(v) = 0\}.$$

Set $g := \dim \Pi^0(G)$.

Let $\pi_* \in \Pi(G)$ and consider a basis $\pi_1, \ldots, \pi_g$ of $\Pi^0(G)$. Define the set

$$\mathcal{P} := \{a \in \mathbb{R}^g : \pi_* + \sum_{x=1}^{g} a_x \pi_x \in \Pi(G)\},$$

which depends on the choice of the functions $\pi_*, \pi_1, \ldots, \pi_g$. This is a polytope.

A result of Shultz shows that all polytopes with rational coefficients can be obtained in this way.[4]

[4] F. W. Shultz: *A characterization of state spaces of orthomodular lattices*. Journal of Combinatorial Theory, Series A, 17(3), 1974.

Polytope compatibility: the general case

Theorem

Let $G = (V, E)$ be a probability hypergraph with associated polytope $\mathcal{P}$. Consider a tuple of $\mathcal{P}$-operators $(B_x)_{x\in[g]} \subseteq \mathcal{M}_d^{sa}(\mathbb{C})$, $d \in \mathbb{N}$ and define

$$\forall v \in V, \qquad A_v := \pi_*(v) I_d + \sum_{x\in[g]} \pi_x(v) B_x \in \mathcal{M}_d^{sa}(\mathbb{C}).$$

Consider also the POVMs $\hat{A}_{\cdot|e} = (A_v)_{v\in e}$ indexed by the hyperedges of G. TFAE:

1. *The $(B_x)_{x\in[g]}$ are $\mathcal{P}$-compatible.*
2. *The POVMs $\hat{A}_{\cdot|e}$ are compatible and there is single POVM $C = (C_\lambda)_{\lambda\in\Lambda}$ s.t.*

$$\forall e \in E, \forall v \in e, \qquad \hat{A}_{v|e} = A_v = \sum_{\lambda\in\Lambda} p(v|e,\lambda) C_\lambda$$

using a post-processing p respecting the symmetry of G:

$$\forall e, f \in E, \forall v \in e \cap f, \forall \lambda \in \Lambda, \qquad p(v|e,\lambda) = p(v|f,\lambda).$$

Summary

- Measurement incompatibility can be phrased as inclusion of free spectrahedra. Base set: diamond.
- Alternatively, measurement incompatibility can be phrased in terms of minimal and maximal matrix convex sets. Base set: cube.
- In both cases, noise robustness corresponds to inclusion constants.
- Generalization: $\mathcal{P}$-compatible operators.
- Examples include magic squares and compatibility with shared elements (under restricted post-processing).
- Polytope compatibility is in one-to-one correspondence to measurement compatibility with shared effects and restricted post-processing.

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[1] AB, I. Nechita: *Joint measurability of quantum effects and the matrix diamond*. Journal of Mathematical Physics, 59, 2018

[2] AB, I. Nechita: *Compatibility of Quantum Measurements and Inclusion Constants for the Matrix Jewel*. SIAM Journal on Applied Algebra and Geometry, 4(2), 2018

Compatibility and matrix convex sets:

[3] AB, I. Nechita: *A tensor norm approach to quantum compatibility*. Journal of Mathematical Physics, 63, 2022

[4] AB, I. Nechita, S. Schmidt: *Polytope compatibility – from quantum measurements to magic squares*. arXiv-preprint arxiv:2304.10920