Matrix convex sets in quantum information theory: Measurement compatibility and beyond

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Matrix convex sets

Matrix convex sets

We consider free sets:

$$\mathcal{F} = \bigsqcup_{i \in \mathbb{N}} \mathcal{F}_i,$$

where $\mathcal{F}_i \subseteq (\mathcal{M}_i^{\mathrm{sa}})^g$.

The free set $\mathcal{F}$ is matrix convex if it is closed under direct sums and unital completely positive maps:

- $(A_1, \ldots, A_g) \in \mathcal{F}_i$, $(B_1, \ldots, B_g) \in \mathcal{F}_j \implies (A_1 \oplus B_1, \ldots, A_g \oplus B_g) \in \mathcal{F}_{i+j}$.
- $(A_1, \ldots, A_g) \in \mathcal{F}_i$, $\Phi : \mathcal{M}_i \to \mathcal{M}_j$ UCP $\implies (\Phi(A_1), \ldots, \Phi(A_g)) \in \mathcal{F}_j$

UCP maps $\Phi : \mathcal{M}_d \to \mathcal{M}_m$ are maps such that $\Phi \otimes \mathrm{id}_n$ is positive for all $n \in \mathbb{N}$ and $\Phi(I_d) = I_m$.

Alternatively, $\Phi(X) = \sum_i K_i^* X K_i$ such that $\sum_i K_i^* K_i = I_m$, $K_i \in \mathcal{M}_{d,m}$.

Minimal and maximal matrix convex sets

- Unless $\mathcal{F}_1$ is a simplex, there are arbitrarily many different matrix convex sets with the same $\mathcal{F}_1$. However, there is a largest and a smallest such set:
- For a closed convex set $\mathcal{C}$,

$$\mathcal{W}_n^{\max}(\mathcal{C}) := \left\{ X \in (\mathcal{M}_n^{\mathrm{sa}})^g : \sum_{i=1}^{g} c_i X_i \leq \alpha I \ \forall (\alpha, c) \text{ supp. hyperplanes for } \mathcal{C} \right\}$$

- For a closed convex set $\mathcal{C}$,

$$\mathcal{W}_n^{\min}(\mathcal{C}) := \left\{ \sum_j X = z_j \otimes Q_j \in (\mathcal{M}_n^{\mathrm{sa}})^g : z_j \in \mathcal{C}, \ Q_j \geq 0 \ \forall j, \sum_j Q_j = I_n \right\}$$

- Observe $\mathcal{W}_1^{\max}(\mathcal{C}) = \mathcal{C} = \mathcal{W}_1^{\min}(\mathcal{C})$. $\mathcal{W}^{\max}(\mathcal{C})$ quantizes hyperplanes, $\mathcal{W}^{\min}(\mathcal{C})$ quantizes extreme points.

Inclusion sets

Definition

Let $d, g \in \mathbb{N}$ and $\mathcal{C} \subset \mathbb{R}^g$ closed convex. The inclusion set is defined as

$$\Delta_{\mathcal{C}}(d) := \left\{ s \in [0,1]^g : s \cdot \mathcal{W}_d^{\max}(\mathcal{C}) \subseteq \mathcal{W}_d^{\min}(\mathcal{C}) \right\}.$$

If $\mathcal{C}$ is the ℓ_∞^g unit ball, we write $\Delta_\square(g,d)$.

Depending on the set $\mathcal{C}$, sometimes bounds on the inclusion set are known.

Measurement compatibility

Quantum states and measurements

- Motivation: Classical state $\rightsquigarrow$ probability distributions: $p \in \mathbb{R}^d$, $p \geq 0$, $\sum_i p_i = 1$.
- Quantum states $\rightsquigarrow$ density matrices: $\rho \in \mathcal{M}_d(\mathbb{C})$, $\rho \geq 0$, $\operatorname{Tr} \rho = 1$.
- Measurement outcomes are labeled $\{1, \ldots, k\}$, need to be assigned probabilities.
- Measurements: Tuples of matrices $(E_1, \ldots, E_k)$ such that $(\operatorname{Tr}[E_1 \rho], \ldots, \operatorname{Tr}[E_k \rho])$ is a probability distribution for all states ρ.
  - $\operatorname{Tr}[E_i \rho] \in \mathbb{R} \rightsquigarrow E_i = E_i^*$.
  - $\operatorname{Tr}[E_i \rho] \geq 0 \rightsquigarrow E_i \geq 0$.
  - $\sum_i \operatorname{Tr}[E_i \rho] = 1 \rightsquigarrow \sum_i E_i = I_d$.
- Tuples of PSD matrices summing to identity are called positive operator-valued measures (POVMs).

Quantum measurements: Compatibility

- Quantum measurements $\rightsquigarrow$ give the probabilities of the classical outcomes when a quantum state enters a measurement apparatus. Mathematically, measurements are modeled by POVMs.

Definition

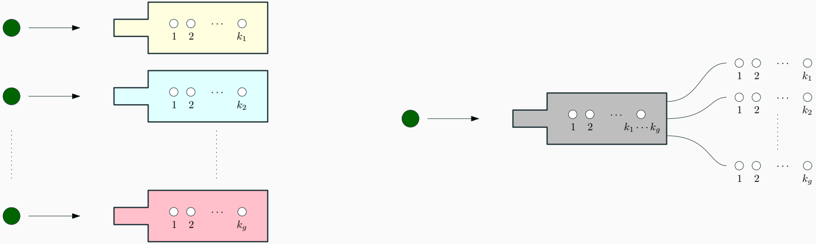
Two POVMs, $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_l)$, are called compatible if there exists a third POVM $C = (C_{ij})_{i \in [k], j \in [l]}$ such that

$$\forall i \in [k], \quad A_i = \sum_{j=1}^{l} C_{ij} \qquad \text{and} \qquad \forall j \in [l], \quad B_j = \sum_{i=1}^{k} C_{ij}.$$

The definition generalizes to g-tuples of POVMs $A^{(1)}, \ldots, A^{(g)}$, having respectively $k_1, \ldots k_g$ outcomes, where the joint POVM C has outcome set $[k_1] \times \cdots \times [k_g]$.

- Other way to say that: jointly measurable.

What does it mean?



- Compatible measurements can be simulated by a single joint measurement, by classically post-processing its outputs $A_i^{(j)} = \sum_\lambda p_j(i|\lambda) C_\lambda$.
- Examples:
  1. Trivial POVMs $A = (p_i I_d)$ and $B = (q_j I_d)$ are compatible.
  2. Commuting POVMs $[A_i, B_j] = 0$ are compatible.
  3. If the POVM A is projective, then A and B are compatible if and only if they commute.

Noisy POVMs

- POVMs can be made compatible by adding noise, i.e. mixing in trivial POVMs.
- Example: dichotomic POVMs and white noise, $s \in [0,1]$:

$$(E, I-E) \mapsto s(E, I-E) + (1-s)(\frac{I}{2}, \frac{I}{2}) \qquad \text{or} \qquad E \mapsto sE + (1-s)\frac{I}{2}.$$

- Taking $s = 1/2$ suffices to render any pair of dichotomic POVMs compatible $\rightsquigarrow$ define $C_{ij} := (E_i + F_j)/4$.
- From now on, we focus on dichotomic (YES/NO) POVMs.

Definition

The compatibility region for g measurements on $\mathbb{C}^d$ is the set

$$\Gamma(g,d) := \{s \in [0,1]^g \ : \ \text{for all quantum effects } E_1, \ldots, E_g \in \mathcal{M}_d(\mathbb{C}),$$
$$\text{the noisy versions } s_i E_i + (1-s_i) I_d/2 \text{ are compatible}\}$$

Compatibility region

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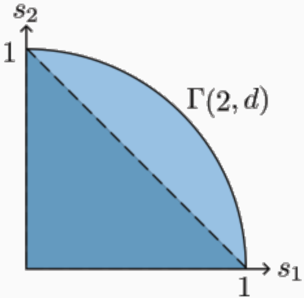
- The set $\Gamma(g,d)$ is convex.
- For all $i \in [g]$, $e_i \in \Gamma(g,d)$: every measurement is compatible with $g-1$ trivial measurements.
- For $d \geq 2$, $(1,1,\ldots,1) \notin \Gamma(g,d)$: there exist incompatible measurements.
- For all $d \geq 2$, $\Gamma(2,d)$ is a quarter-circle.



Generally speaking, the set $\Gamma(g,d)$ tells us how robust (to noise) is the incompatibility of g dichotomic measurements on $\mathbb{C}^d$.

Link measurement compatibility and matrix convex sets

Measurement compatibilty revisited

From now on, we concentrate on measurements with two outcomes and identify $E^{(i)} = \{E_i, I - E_i\}$ with E_i.

Theorem

Let

$$A = \sum_{j=1}^{g} e_j \otimes (2E_j - I).$$

Then,

1. $A \in \mathcal{W}_d^{\max}(\mathcal{B}(\ell_\infty^g))$ *if and only if* $\{E_j\}_{j\in[g]}$ *is a collection of POVMs.*
2. $A \in \mathcal{W}_d^{\min}(\mathcal{B}(\ell_\infty^g))$ *if and only if* $\{E_j\}_{j\in[g]}$ *is a collection of compatible POVMs.*

Proof sketch

- $\mathcal{W}_d^{\max}(\mathcal{B}(\ell_\infty^g))$ is given in terms of hyperplanes. Have to verify $-I \leq A_i = 2E_i - I \leq I$ $\implies 0 \leq E_i \leq I$.
- Reminder:

$$\mathcal{W}_n^{\min}(\mathcal{B}(\ell_\infty^g)) := \left\{ X = \sum_j z_j \otimes Q_j \in (\mathcal{M}_n^{\mathrm{sa}})^g : z_j \in \mathcal{C}\ \forall j,\ Q \text{ POVM} \right\}.$$

- Going to extreme points:

$$2E_j - I = \sum_{\epsilon \in \{\pm 1\}} \epsilon(j) Q_\epsilon.$$

- Using $\sum_\epsilon Q_\epsilon = I$:

$$E_j = \sum_{\epsilon \in \{\pm 1\}} \delta_{\epsilon(j),1} Q_\epsilon.$$

- $\{Q_\epsilon\}_\epsilon$ is a joint POVM.

Inclusion sets and compatibility regions

Theorem

Let $g, d \in \mathbb{N}$. Let $s \in [0,1]^g$. Then, $\{s_i E_i + (1-s_i)I/2\}_{i\in[g]}$ is a collection of compatible POVMs for all POVMs $\{E_i\}_{i\in[g]}$, if and only if $s \in \Delta_\square(g,d)$. An equivalent way to phrase this is $\Gamma(g,d) = \Delta_\square(g,d)$.

- This follows from the computation

$$A_i' = 2(s_i E_i + (1-s_i)I/2) - I = s_i(2E_i - I) = s_i A_i.$$

- So adding noise means scaling the tensor A and hence $s \cdot \mathcal{W}_d^{\max}(\mathcal{B}(\ell_\infty^g))$ is the set of noisy measurements.
- Thus, $s \cdot A \in \mathcal{W}_d^{\min}(\mathcal{B}(\ell_\infty^g))$ means the noisy measurements are compatible.

Polytope compatibility

Work in progress with Ion Nechita and Simon Schmidt

Polytope compatibility

Definition

Let $\mathcal{P}$ be a polytope in $\mathbb{R}^g$ such that $0 \in \operatorname{int} \mathcal{P}$. Let

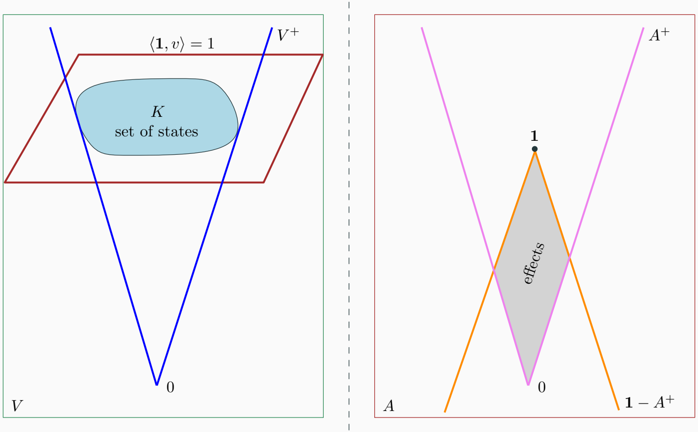
$$A = (A_1, \ldots, A_g) \in \mathcal{M}_d^{\mathrm{sa}}(\mathbb{C})^g \cong \mathbb{R}^g \otimes \mathcal{M}_d^{sa}(\mathbb{C})$$

a g-tuple of Hermitian matrices. Then, A are $\mathcal{P}$-*operators* if and only if $A \in \mathcal{W}_d^{\max}(\mathcal{P})$. Moreover, A are $\mathcal{P}$-*compatible* if and only if $A \in \mathcal{W}_d^{\min}(\mathcal{P})$.

Motivation:

- A are $\mathcal{B}(\ell_\infty^g)$-operators if and only if $\frac{1}{2}(A_i + I)$ are dichotomic POVMs.
- A are $\mathcal{B}(\ell_\infty^g)$-compatible if and only if $\frac{1}{2}(A_i + I)$ are compatible dichotomic POVMs.

Interlude: General Probabilistic Theories



- A GPT is a triple $(V, V^+, \mathbb{1})$, where V is a vector space, $V^+ \subseteq V$ is a cone, and $\mathbb{1}$ is a linear form on V; $A = V^*$, $A^+ = (V^+)^*$, and $\mathbb{1} \in A^+$
- The set of states $K := V^+ \cap \mathbb{1}^{-1}(\{1\})$

Equivalent formulation

Theorem

Let d, g, $k \in \mathbb{N}$ and let $\mathcal{P}$ be a polytope with k extremal points $v_1, \dots, v_k \in \mathbb{R}^g$ such that $0 \in \operatorname{int} \mathcal{P}$. Let $A = (A_1, \dots, A_g) \in \mathcal{M}_d^{\mathrm{sa}}(\mathbb{C})^g$ be a g-tuple of Hermitian matrices. Let us consider the map $\mathcal{A} : \mathcal{M}_d^{\mathrm{sa}} \to \mathbb{R}^g$,

$$\mathcal{A}(X) = (\mathrm{Tr}[A_1 X], \dots, \mathrm{Tr}[A_g X]).$$

Then,

1. *A are $\mathcal{P}$-operators if and only if $\mathcal{A}$ is a channel between $(\mathcal{M}_d^{\mathrm{sa}}, \mathrm{PSD}_d, \mathrm{Tr})$ and $(V(\mathcal{P}), V(\mathcal{P})^+, \mathbb{1}_{\mathcal{P}})$.*
2. *A are $\mathcal{P}$-compatible if and only if in addition $\mathcal{A}$ factors through the k-simplex Δ_k.*

Interpretation: $\mathcal{P}$ defines some kind of allowed post-processing.

In the case of $\mathcal{B}(\ell_\infty^g)$: Classical post-processing.

Magic squares

A magic square is a collection of positive operators A_{ij}, $i, j \in [N]$, such that

$$\begin{array}{ccccccccc} A_{11} & + & A_{12} & + & \dots & + & A_{1N} & = I \\ + & & + & & & & + & \\ \vdots & & \vdots & & \ddots & & \vdots & \vdots \\ + & & + & & & & + & \\ A_{N1} & + & A_{N2} & + & \dots & + & A_{NN} & = I \\ \| & & \| & & & & \| & \\ I & & I & & \dots & & I & \end{array}$$

The magic square is said to be semiclassical if

$$A = \sum_{i,j \in [N]} |i\rangle\langle j| \otimes A_{ij} = \sum_{\pi \in \mathcal{S}_N} P_\pi \otimes Q_\pi,$$

where P_π is the permutation matrix associated to π and $\{Q_\pi\}_\pi$ is a POVM.

Birkhoff polytope compatibility

Definition

For a given $N \geq 2$, the Birkhoff body $\mathcal{B}_N(1)$ is defined as the set of $(N-1) \times (N-1)$ truncations of $N \times N$ bistochastic matrices, shifted by J/N:

$$\mathcal{B}_N = \{A^{(N-1)} - J_{N-1}/N \;:\; A \in \mathcal{M}_N(\mathbb{R}) \text{ bistochastic}\} \subset \mathbb{R}^{(N-1)^2}.$$

Theorem

Consider a $(N-1)^2$-tuple of selfadjoint matrices $A \in \mathcal{M}_d^{\mathrm{sa}}(\mathbb{C})^{(N-1)^2}$ and the corresponding matrix $\tilde{A} \in \mathcal{M}_N(\mathcal{M}_d(\mathbb{C}))$. Then:

1. *The matrix $\tilde{A}$ is a magic square if and only if $A - I/N$ are $\mathcal{B}_N$-operators.*
2. *The matrix $\tilde{A}$ is a semiclassical magic square if and only if $A - I/N$ are $\mathcal{B}_N$-compatible.*

Relation to measurement incompatibility

Is being a semiclassical magic square the same as being compatible? No.

$\frac{1}{2}\|0\rangle\langle 0\|$	$\frac{1}{2}\|1\rangle\langle 1\|$	0	$\frac{1}{2}I_2$
$\frac{1}{2}\|1\rangle\langle 1\|$	$\frac{1}{2}\|0\rangle\langle 0\|$	$\frac{1}{2}I_2$	0
0	$\frac{1}{2}I_2$	$\frac{1}{2}\|+\rangle\langle +\|$	$\frac{1}{2}\|-\rangle\langle -\|$
$\frac{1}{2}I_2$	0	$\frac{1}{2}\|-\rangle\langle -\|$	$\frac{1}{2}\|+\rangle\langle +\|$

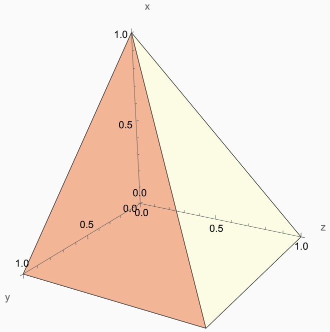
These measurements are compatible, but they do not form a semiclassical magic square.

Reason: $\mathcal{B}_N$-compatibility restricts the post-processing to $p_i(j|\lambda) = p_j(i|\lambda)$, i.e., enforces special structure in the joint POVM.

Measurement compatibility with shared effects

Can we generalize the magic square example?

$$\mathcal{P} = (-1/3, -1/3, -1/3) + \text{conv}\{((1,0,0),(0,0,0),(0,0,1),(0,1,0),(0,1,1)\}.$$



Consider $(A, B, C) \in (\mathcal{M}_d(\mathbb{C})^{sa})^3$. Then, we have $(A, B, C) + 1/3(I, I, I) \in \mathcal{W}_d^{\max}(\mathcal{P})$ if and only if both $(A, B, I_d - A - B)$ and $(A, C, I_d - A - C)$ are POVMs.

Measurement compatibility with shared effects, continued

When does $(A, B, C) + 1/3(I, I, I) \in \mathcal{W}_d^{\min}(\mathcal{P})$ hold? Equivalent to the existence of a joint measurement such that

Q_1	0	0	$= A$
0	Q_5	Q_4	$= B$
0	Q_3	Q_2	$= I_d - A - B$
$= A$	$= C$	$= I_d - A - C$	

Not all joint measurements are of this form, check

$$\left(\frac{1}{2} I_2, \frac{1}{2} |0\rangle\langle 0|, \frac{1}{2} |1\rangle\langle 1|\right) \qquad \text{and} \qquad \left(\frac{1}{2} I_2, \frac{1}{2} |+\rangle\langle +|, \frac{1}{2} |-\rangle\langle -|\right).$$

Summary

- Measurement incompatibility can be phrased as inclusion of matrix convex sets. Base set: cube.
- Noise robustness corresponds to inclusion constants.
- Generalization: $\mathcal{P}$-operators and $\mathcal{P}$-compatible operators.
- Examples include magic squares and compatibility with shared elements (under restricted post-processing).

Can we find more tasks in quantum information theory which can be formulated as $\mathcal{P}$-compatibility?