Lossless quantum compression of quantum measurements

Joint work with Michael M. Wolf and Lukas Rauber

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Bremen, 14. März 2017

Measurements

Aim of this talk: Compression of *Hilbert space dimension*

Measurement upon preparation $\rho \in \mathcal{S}(\mathbb{C}^D)$:

- Measurement outcomes $\{ a_i \}_{i=1}^k$, probabilities $\{ p_i \}_{i=1}^k$
- Effect operators:

$$\mathcal{E}(\mathbb{C}^D) = \{ E \in \mathcal{M}_D : 0 \leq E \leq \mathbb{1} \}$$

- Associate with probability:

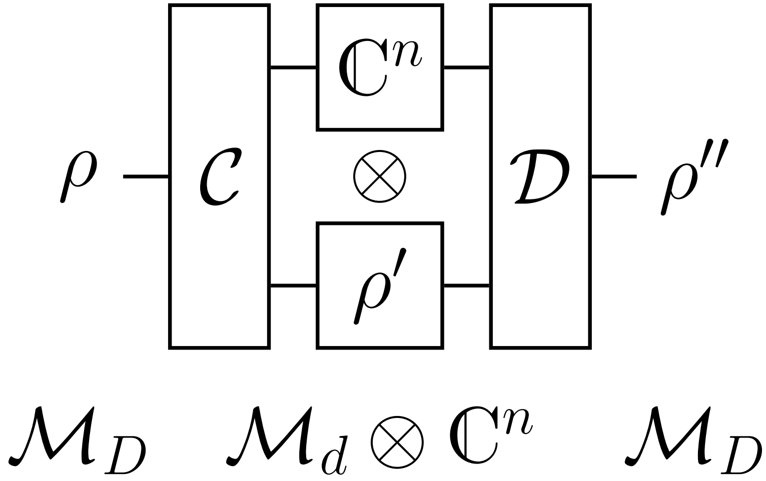
$$p_i = \mathrm{Tr}\,(E_i \rho) \qquad \forall i \in \{ 1, \ldots, k \}$$

- Normalization:

$$\mathbb{1} = \sum_{i=1}^k E_i$$

From now on: Measurement = Set of effect operators

Setup



- $\mathcal{C}$, $\mathcal{D}$ completely positive trace preserving maps, $\mathcal{T} = \mathcal{D} \circ \mathcal{C}$
- Fixed set $\mathcal{O} \subset \mathcal{E}(\mathbb{C}^D)$ of effect operators
- Constraints:

$$\mathrm{Tr}\left(\rho E\right) = \mathrm{Tr}\left(\mathcal{T}(\rho)E\right) \qquad E \in \mathcal{O}$$

- Require this to hold *for all* $\rho \in \mathcal{S}(\mathbb{C}^D)$
- Aim: Find *d* as small as possible, *n* may be arbitrarily large

Generic case

Fact:

Let $A, B \in \mathcal{M}_D$. Generically, $\mathrm{C}^*(A, B) = \mathcal{M}_D$.

Theorem (Generic lower bound)

Let $\mathcal{O}$ be a set of effect operators such that $\mathrm{C}^*(\mathcal{O}) = \mathcal{M}_D$. Then the effect operators are incompressible and furthermore $\mathcal{T} = \mathrm{id}$ holds.

In a physical setup, typically more structure $\rightarrow$ subalgebras

Subalgebras

Now, $\mathrm{C}^*(\mathcal{O}) = \bigoplus_{i=1}^{s} \mathcal{M}_{D_i}$.

Theorem (Lower/Upper bound subalgebras)

Let $\mathcal{O}$ be a set of effect operators such that $\mathrm{C}^*(\mathcal{O}) = \bigoplus_{i=1}^{s} \mathcal{M}_{D_i}$ and $\sum_{i=1}^{s} D_i = D$. Then $\min_{i\in[s]} D_i$ is a lower bound for the compression dimension d. Furthermore, the compression dimension is upper bounded by $\max_{i\in[s]} D_i$.

Explicit compression channel: Use block structure of effects, classical side information contains block number.

Example (Two bipartite von Neumann measurements)

Let $P, Q \in \mathcal{E}(\mathbb{C}^D)$ be two orthogonal projections. Then, $\mathcal{O} := \{ P, \mathbb{1} - P, Q, \mathbb{1} - Q \}$ has compression dimension at most 2.

Computing the dimension

What can we say if for $\mathrm{C}^*(\mathcal{O}) = \bigoplus_{i=1}^{s} \mathcal{M}_{D_i}$ upper and lower bound do not match?

- Can phrase this as an interpolation problem: Check if there is a channel $\Phi : \mathcal{M}_D \mapsto \mathcal{M}_{D_1}$ such that

$$E^1 = \Phi\left(\begin{bmatrix} 0 & & & \\ & E^2 & & \\ & & \ddots & \\ & & & E^s \end{bmatrix}\right) \qquad \forall E \in \mathcal{O}.$$

- This can be checked using a semidefinite program by a result of Heinosaari et al.[1]
- Repeat for every block from largest to smallest until interpolation map no longer exists

[1] T. Heinosaari, M. A. Jivulescu, D. Reeb and M. M. Wolf. Extending quantum operations. *Journal of Mathematical Physics, 53(10):102208, 2012.*

Proof techniques

Constraints:

$$\mathrm{Tr}\left(\rho E\right) = \mathrm{Tr}\left(\mathcal{T}(\rho) E\right) = \mathrm{Tr}\left(\rho \mathcal{T}^*(E)\right) \qquad \forall \rho \in \mathcal{S}(\mathbb{C}^D), E \in \mathcal{O}$$

Several ways to prove bounds on compression dimension:

- Algebraic: Use some result by Arveson[2] about fixed points of completely positive maps of norm at most 1.
- Geometric: Reduce problem to determining whether $\|E_1 + tE_2\|_\infty = \|\mathcal{D}^*(E_1) + t\mathcal{D}^*(E_2)\|_\infty \; \forall t \in \mathbb{R}$ and for $E_1, E_2 \in \mathcal{O}$
  Bézout's theorem: Not possible if E_1, E_2 give rise to an irreducible characteristic polynomial
- Algebraic argument stronger statement, geometric more widely applicable
- Example: Several copies of one state

[2] William Arveson: Subalgebras of C*-algebras II. *Acta Mathematica*, 128(1):271–308, 1972.

Conclusion

- Generically no compression possible with respect to Hilbert space dimension
- Physical setups: Dimension of largest block achievable, dimension of smallest block lower bound
- Compression dimension can be checked algorithmically