Local factorisation for the dynamics of quantum spin system

Andreas Bluhm

Technical University of Munich

August 22, 2016

Table of contents

Area laws on gapped quantum spin systems

Area law

$$S(\rho_X) \leq C|\partial X| + D$$

Example (Maximally entangled state)

$$|\psi\rangle = \sum_{\alpha=1}^{N^{|X|}} \frac{1}{\sqrt{N^{|X|}}} |\Psi_{X,\alpha}\rangle \otimes |\Psi_{X^c,\alpha}\rangle$$

$$S(\rho_X) = |X| \ln N$$

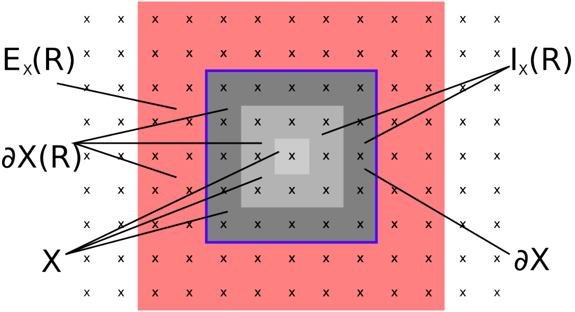
One-dimensional case proved by Hastings[1]

[1] M. B. Hastings, J. Stat. Mech. (2007), P08024.

Stability of the area law

The lattice

- Lattice: Set $\mathbb{Z}^\nu$, graph distance d
- Hilbert spaces: $\sup_{x\in\Gamma} \dim \mathcal{H}_x \leq N$



- Quasi-local algebra $\mathcal{A}$ and integrable underlying structure
- Local normality: For any $A \in \mathcal{A}_X$, we have $\omega(A) = \mathrm{Tr}\left(\rho_X^\omega A\right)$.

The algebra

- $\mathcal{A}_X = \mathcal{B}(\mathcal{H}_X)$
- Quasi-local algebra:

$$\mathcal{A}_{\mathrm{loc}} = \bigcup_{\substack{Z\subset\Gamma \\ |Z|<\infty}} \mathcal{A}_Z; \quad \mathcal{A} = \overline{\mathcal{A}_{\mathrm{loc}}}^{\|\cdot\|}$$

- For $X \subset Y$, $A \in \mathcal{A}_X$: Identify A with $A \otimes \mathbb{1}_{Y\setminus X} \in \mathcal{A}_Y$.
- Local normality: $A \in \mathcal{A}_X$

$$\omega(A) = \mathrm{Tr}\left(\rho_X^\omega A\right)$$

Integrable underlying structure

Assume $\exists F : [0, \infty) \to (0, \infty)$ non-increasing s.t.:

1. Uniform integrability:

$$\|F\| = \sup_{x \in \Gamma} \sum_{y \in \Gamma} F(d(x, y)) < \infty$$

2. Convolution condition: $\exists C_F > 0$ such that

$$\sum_{z \in \Gamma} F(d(x, z)) F(d(z, y)) \leq C_F F(d(x, y)).$$

Example (in $\mathbb{Z}^\nu$ [2])

$$F_\nu(r) = (1 + r)^{-(\nu+\epsilon)}, \quad \epsilon > 0$$

[2] B. Nachtergaele, Y. Ogata and R. Sims, J. Stat. Phys. 124(1), 1–13 (2006).

The spectral flow

- Projection onto lower part of spectrum[3]:

$$P_\Lambda(s) = U_\Lambda(s) P_\Lambda(0) U_\Lambda(s)^*$$

- Spectral flow: $\alpha_s^\Lambda(A) = U_\Lambda(s)^* A U_\Lambda(s)$, $A \in A_\Lambda$
- Generator:

$$\frac{d}{ds} U_\Lambda(s) = i D_\Lambda(s) U_\Lambda(s), \quad U_\Lambda(0) = \mathbb{1}$$

with

$$D_\Lambda(s) = \sum_{Z \subset \Lambda} \Psi_\Lambda(Z, s)$$

[3] S. Bachmann, S. Michalakis, B. Nachtergaele and R. Sims, Comm. Math. Phys. 309, 835-871 (2012).

Lieb-Robinson bounds

Theorem (LR-bounds for the spectral flow[4])

Let $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$ and $s \in [0,1]$. Then

$$\left\| \left[\alpha_s^\Lambda(A), B \right] \right\| \leq 2 \|A\| \, \|B\| \, K_\Psi \left(e^{v_\Psi s} - 1 \right) \min\{ \; |X|; |Y| \, \} \, u(R)$$

for $d(X,Y) = R > 0$, $u(R)$ almost exponential and $K_\Psi, v_\Psi > 0$.

[4] S. Bachmann, S. Michalakis, B. Nachtergaele and R. Sims, Comm. Math. Phys. 309, 835-871 (2012).

Local factorisation theorem

Theorem

For a finite $X \subset \Gamma$ as before, $R \in \mathbb{N}$, and any finite $\Lambda \subset \Gamma$ such that $\overline{X}^R \subset \Lambda$,

$$\left\| U_\Lambda(t,s) - \left(U_X(t,s) \otimes U_{\Lambda\setminus X}(t,s)\right) \widetilde{U}_{\partial_R X}(t,s)^* \right\|$$
$$\leq |\partial X| \, \|\Psi\|_\xi \, \|F\| \, |t-s| \left[2\zeta(R/2) + |\partial X| \, \kappa (R/2)^{2d} \xi(R/2)(e^{v_\xi |t-s|} - 1) \right]$$

where $\kappa = \frac{G^2 \|F_\xi\|}{C_\xi}$.

Approximate P_0

Theorem

Consider $\mathbb{Z}^\nu$ with graph distance d. Let Φ be an interaction which is bounded, finite ranged and generates a finite-volume Hamiltonian with unique ground state on $\Lambda \subset \mathbb{Z}$ finite. Then for any $l > \mathcal{R}$ there exist two projections $P_X \in \mathcal{A}_X$ and $P_{X^c} \in \mathcal{A}_{\Lambda \setminus X}$ and an observable $P_{X_{\mathrm{bd}}} \in \mathcal{A}_{X_{\mathrm{bd}}(3l)}$ with $\|P_{X_{\mathrm{bd}}}\| \leq 1$ such that

$$\|P_{X_{\mathrm{bd}}} P_X P_{X^c} - P_0\| \leq K C_\Phi N_\Phi |\partial_{\mathcal{R}} X|^2 l^{\frac{7}{2}} e^{-\frac{l}{2\xi}} =: \tilde{\epsilon}(l)$$

with

$$\frac{2}{\xi} = \frac{\mu\gamma^2}{\mu^2 v^2 + \gamma^2}.$$

Quasi-local Hamiltonians with spectral gap

- Family of Hamiltonians ($s \in [0,1]$):

$$H_\Lambda(s) = \sum_{Z \subset \Lambda} \Phi(Z, s)$$

- Quasi-local interaction $\Phi(Z, s) \in \mathcal{A}_Z$
- Gap:

$$\sup \{ \ |\lambda_1 - \lambda_2| : \lambda_1 \in \sigma_1(H_\Lambda(s)), \lambda_2 \in \sigma_2(H_\Lambda(s)) \ \} \geq \gamma > 0$$

Decay of the eigenvalues

Lemma (Decay of the perturbed eigenvalues[5])

Let $\omega \circ \alpha_s^{\mathbb{Z}^\nu} \in \mathcal{S}(\mathcal{A})$ *with associated density matrix* $\rho_X^\omega(s)$, $R \geq R_2$. *Then*

$$\sum_{\alpha = N^{G_3 R^\nu |\partial X|}+1}^{q_\omega(s)} \sigma_\alpha^\omega(s) \leq \sum_{\alpha = N^{\kappa R^\nu |\partial X|}+1}^{q_\omega(0)} \sigma_\alpha^\omega(0) + 2\epsilon(R, s)$$

with $G_3 > 0$, $\kappa > 0$.

$$2\epsilon(R_0, s) \leq \epsilon \rightarrow S\left(\rho_X^\omega(s)\right) \leq C R_0^\nu |\partial X| + D$$

[5] S. Michalakis, Arxiv:1206.6900v2 (July 2012).

Approximation: Ideas of the proof

- Definition of the unitary: $\tilde{U}_{\partial X(2R)}(0) = \mathbb{1}$

$$\frac{d}{ds}\tilde{U}_{\partial X(2R)}(s) = -i\alpha_s^{\partial X(2R)}\left(\widetilde{D}_{\partial X(R)}(s)\right)\tilde{U}_{\partial X(2R)}(s)$$

- Inner part:

$$\widetilde{D}_{\partial X(R)}(s) = \sum_{\substack{Z \subset \partial X(R): \\ Z \cap X \neq \emptyset, Z \cap X^c \neq \emptyset}} \Psi(Z, s)$$

- Define: $V(s) = U_\Lambda(s)^* U_X(s) \otimes U_{X^c}(s)$
- As differential equation:

$$\frac{d}{ds}V(s) = -i\alpha_s^\Lambda(D_\Lambda - D_X - D_{X^c})V(s), \quad V(0) = \mathbb{1}$$

- Norm grows with $|\partial X|$: $\left\|\widetilde{D}_{\partial X(R)}(s)\right\| \leq \|\Psi\|_{F_\Psi} \|F_\Psi\| \, |\partial X| \kappa R^\nu$